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Transverse (Betatron) Motion

Linear betatron motion
Effects of imperfections of magnets
Dispersion function of off momentum particle
Simple Lattice design considerations

Longitudinal Motion

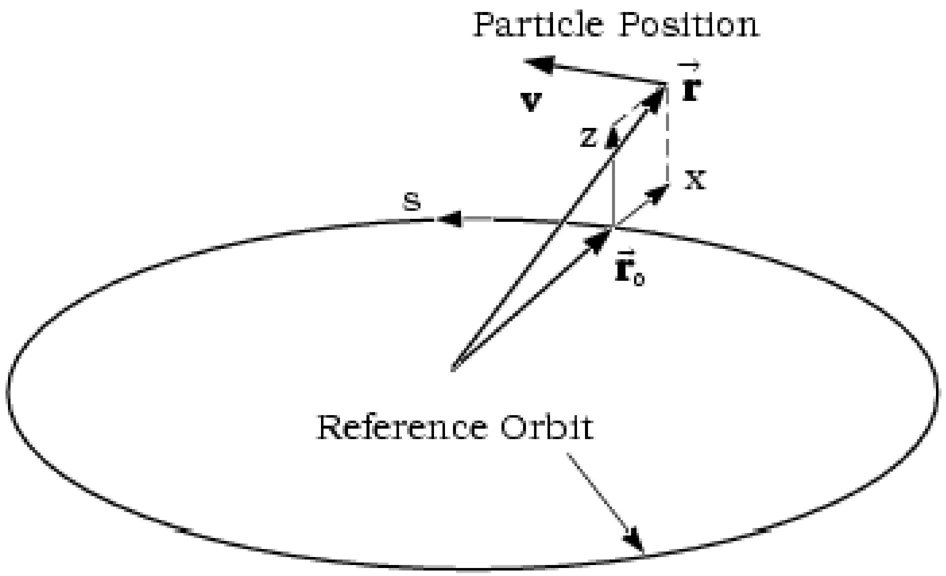
Equation of longitudinal motion
Phase stability
Adiabatic synchrotron motion

Update is available at:
http://physics.indiana.edu/~shylee/ocpa10/

$$\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{\upsilon} \times \vec{B})$$

$$\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}, \ \vec{B} = \nabla \times \vec{A}$$

$$H = e\Phi + c\sqrt{m^2c^2 + (\vec{p} - e\vec{A})^2}$$



Frenet-Serret coordinate system: We assume that there exists a closed orbit $r_0(s)$. The coordinates around the reference orbit is defined by

$$\hat{s} = \frac{d\vec{r}_0}{ds}, \quad \hat{x} = -\rho \frac{d\hat{s}}{ds}, \quad \hat{z} = \hat{x} \times \hat{s}$$

$$\vec{r} = \vec{r}_0 + x\hat{x} + z\hat{z} \qquad \hat{x}' = \tfrac{1}{\rho}\hat{s} + \tau\,\hat{z}, \quad \hat{s}' = -\tau\,\hat{x}$$

How to transform from the original coordinate system onto the Frenet-Serret coordinate system? Generating function!

$$F_3(P, x, s, z) = -\vec{P} \cdot (\vec{r}_0 + x\hat{x} + z\hat{z})$$

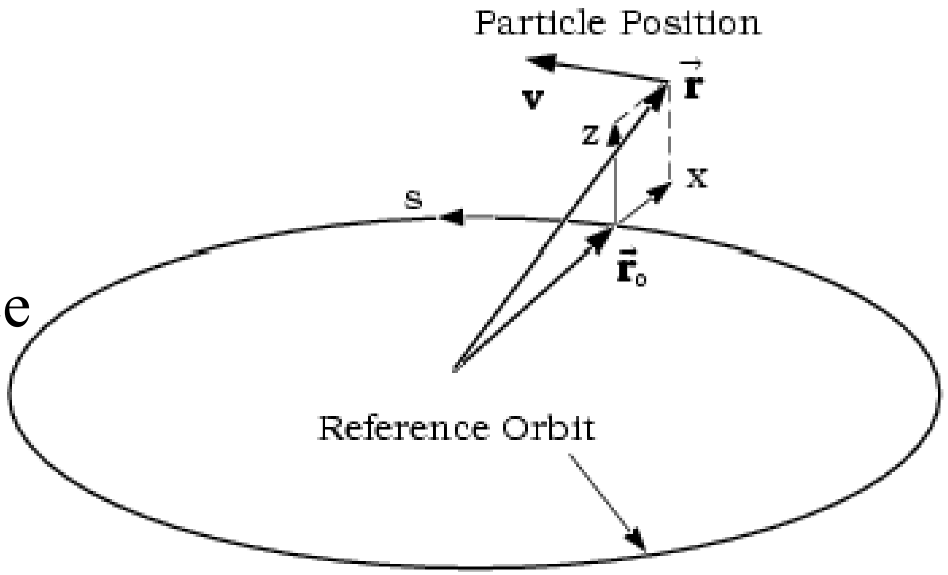
$$p_s = -\frac{\partial F_3}{\partial s} = (1 + \frac{x}{\rho})\vec{P} \cdot \hat{s}, \ p_x = -\frac{\partial F_3}{\partial x} = \vec{P} \cdot \hat{x}, \ p_z = -\frac{\partial F_3}{\partial z} = \vec{P} \cdot \hat{z},$$

$$A_s = (1+\frac{x}{\rho})\vec{A}\cdot\widehat{s},\;\; A_x = \vec{A}\cdot\widehat{x},\;\; A_z = \vec{A}\cdot\widehat{z},$$

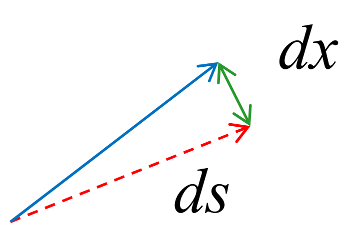
$$H = e\Phi + c\left[m^2c^2 + \frac{(p_s - eA_s)^2}{(1+x/\rho)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2\right]^{1/2}$$

$$\dot{s} = \frac{\partial H}{\partial p_s}, \dot{p}_s = -\frac{\partial H}{\partial s}; \quad \dot{x} = \frac{\partial H}{\partial p_x}, \dot{p}_x = -\frac{\partial H}{\partial x}; \quad \dot{z} = \frac{\partial H}{\partial p_z}, \dot{p}_z = -\frac{\partial H}{\partial z}.$$

The phase space coordinates are **(*x,s,z*)** with independent coordinate ***t***. In one revolution, the time advances T_0, called the orbital period. In one orbital period, the particle orbit is equal to the circumference *C*. All accelerator components repeat in each orbital period. It would be nice to use ***s*** as the independent coordinate. How to make this coordinate transfer?



$$x' = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial H}{\partial p_s}\right)^{-1} = \frac{\partial(-p_s)}{\partial p_x},$$



$$dH = (\partial H/\partial p_x)dp_x + (\partial H/\partial p_s)dp_s = 0$$

$$t' = \frac{\partial p_s}{\partial H}, H' = -\frac{\partial p_s}{\partial t}; \quad x' = -\frac{\partial p_s}{\partial p_x}, p_x' = \frac{\partial p_s}{\partial x}; \quad z' = -\frac{\partial p_s}{\partial p_z}, p_z' = \frac{\partial p_s}{\partial z}.$$

These equations indicate that $-p_s$ becomes the new Hamiltonian with the $(x,p_x,z,p_z,t,-H)$ and s as the independent coordinate.

$$\tilde{H} = -\left(1+\frac{x}{\rho}\right)\left[\frac{(H-e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2\right]^{1/2} - eA_s,$$

$$\tilde{H} \approx -p\left(1+\frac{x}{\rho}\right) + \frac{1+x/\rho}{2p}[(p_x - eA_x)^2 + (p_z - eA_z)^2] - eA_s$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

$$x' = \frac{\partial \tilde{H}}{\partial p_x}, \quad p_x' = -\frac{\partial \tilde{H}}{\partial x}, \quad z' = \frac{\partial \tilde{H}}{\partial p_z}, \quad p_z' = -\frac{\partial \tilde{H}}{\partial z}, \quad \boxed{t' = \frac{\partial \tilde{H}}{\partial H}, \quad -H' = -\frac{\partial \tilde{H}}{\partial t}.}$$

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s), \quad \phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2 E}\Delta E_{n+1}$$ Synchrotron motion

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$ Hill's equation

Transverse magnetic field: $\nabla\times A=B,\ \nabla\bullet B=0$. For 2D magnetic field, B can be represented by either one component of the vector potential A_s, or by a scaler potential Φ, i.e. $B_x=-\partial A_s/\partial z,\ B_z=\partial A_s/\partial x$, or $B_x=\nabla_x\Phi,\ B_z=\nabla_z\Phi$. Although the field can be represented two ways, only the vector potential serves as the "potential" in the betatron Hamiltonian. For two dimensional magnetic field, one can expand the magnetic field using **Beth representation**:

$$H=-p(1+\frac{x}{\rho})+\frac{1+x/\rho}{2p}[p_x^2+p_z^2]-eA_s$$

$$x''-\frac{\rho+x}{\rho^2}=\pm\frac{B_z}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2,$$

$$z''=\mp\frac{B_x}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2.$$

$$x''+K_x(s)x=\frac{\Delta B_z}{B\rho},$$

$$z''+K_z(s)z=-\frac{\Delta B_x}{B\rho}$$

$$B_z+jB_x=B_0\sum_n(b_n+ja_n)(x+jz)^n,$$

$$A_s=\mathrm{Re}\left\{B_0\sum_n\frac{1}{n+1}(b_n+ja_n)(x+jz)^{n+1}\right\}$$

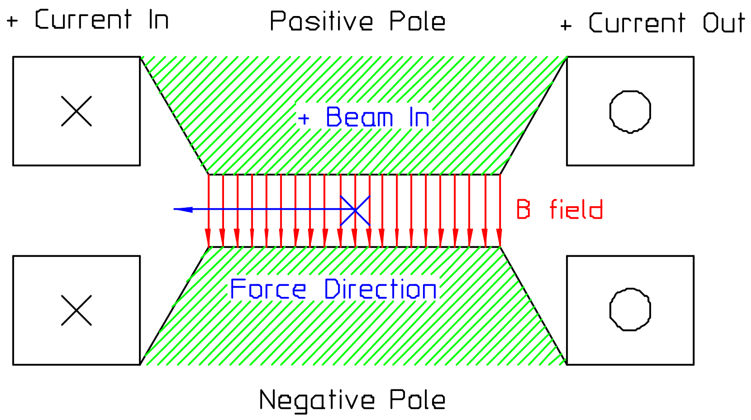
b_0 : dipole, $B_z=B_0b_0$,

b_1 : quad, $B_z=B_0b_1x$, $B_x=B_0b_1z$,

a_0 : skew (vertical) dipole; $B_x=B_0a_0$,

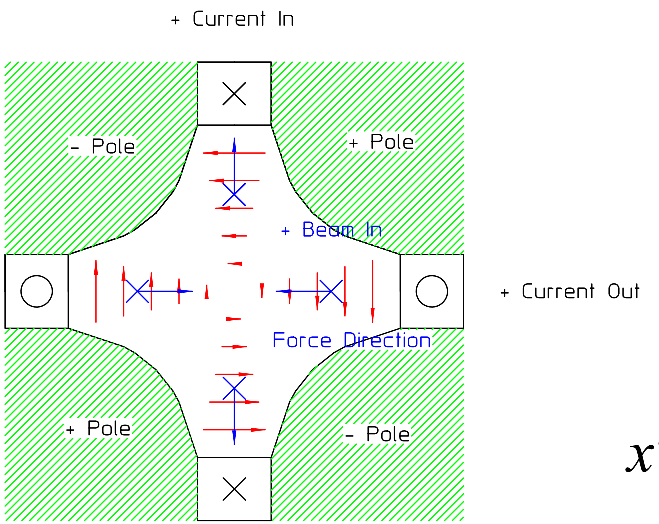
a_1 : skew quad; $B_z=-B_0a_1z$, $B_x=B_0a_1x$,

b_2 : sextupole, a_2 : skew sextupole;



$$B_0 = \frac{\mu_0 NI}{g}, \qquad L = \frac{\mu_0 N^2 \ell w}{g}$$

$$A_s = B_0 x$$



$$B_1 = \frac{2\mu_0 NI}{a^2}, \qquad L = \frac{8\mu_0 N^2 \ell x_c^2}{a^2}$$

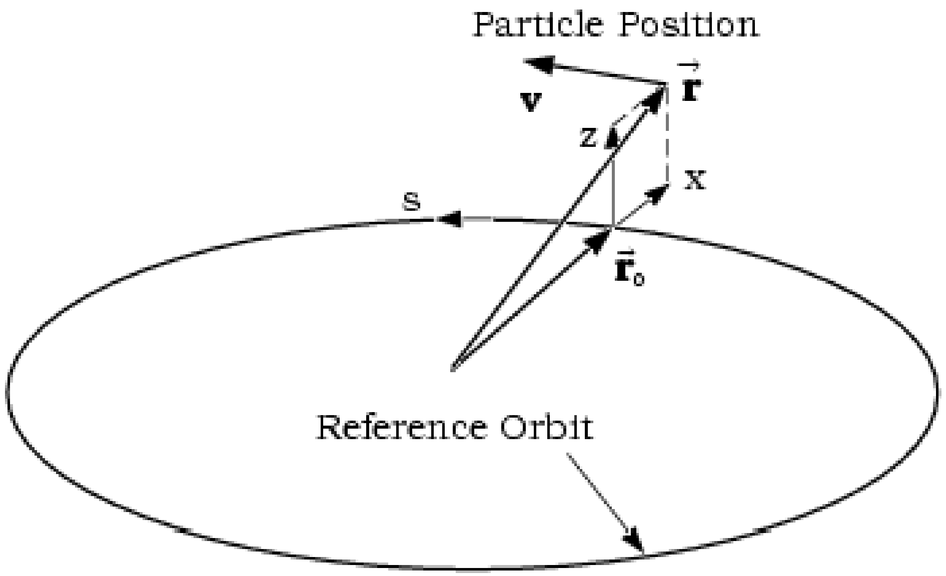
$$A_s = \frac{B_1}{2}\left(x^2 - z^2\right)$$

$$x'' + K_x(s)x = \pm\frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = \mp\frac{\Delta B_x}{B\rho}$$

$$K_x(s) = \frac{1}{\rho^2} \mp \frac{B_1}{B\rho}, \quad K_z(s) = \pm\frac{B_1}{B\rho}$$

The upper/lower sign for positive/negative charge particles

How to solve the Hill's equation?



$$x'' + K_x(s)x = 0, \quad z'' + K_z(s)z = 0$$

Let y represent x or z:

$$y'' + K(s)y = 0,$$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = M(s, s_0) \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

The focusing function is piecewise constant!

$$K(s) = \begin{cases} K > 0 \\ K < 0 \\ 0 \end{cases} \quad y = \begin{cases} A \sin\sqrt{K}s + B\cos\sqrt{K}s \\ A\sinh\sqrt{|K|}s + B\cosh\sqrt{|K|}s \\ A + Bs \end{cases}$$

$$\boxed{\mathrm{Det}(M(\mathrm{s}_2,\mathrm{s}_1))=1}$$

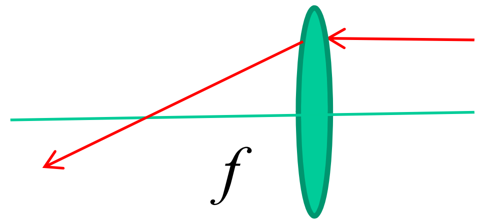
$$M(s, s_0) = \begin{pmatrix} \cos\sqrt{K}(s-s_0) & \frac{1}{\sqrt{K}}\sin\sqrt{K}(s-s_0) \\ -\sqrt{K}\sin\sqrt{K}(s-s_0) & \cos\sqrt{K}(s-s_0) \end{pmatrix}, \quad \ldots$$

$$y'' + K(s)y = 0, \qquad K_x(s) = \frac{1}{\rho^2} \mp \frac{1}{B\rho}\frac{\partial B_z}{\partial x}, \quad K_z(s) = \pm \frac{1}{B\rho}\frac{\partial B_z}{\partial x},$$

Thin lens approximation: Let $|K|\ell \to 1/f$ as $\ell \to 0$.

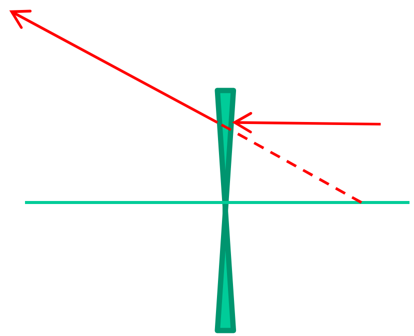
1. focusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$



2. de-focusing quadrupole:

$$M(s,s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}$$



3. Drift space: K=0

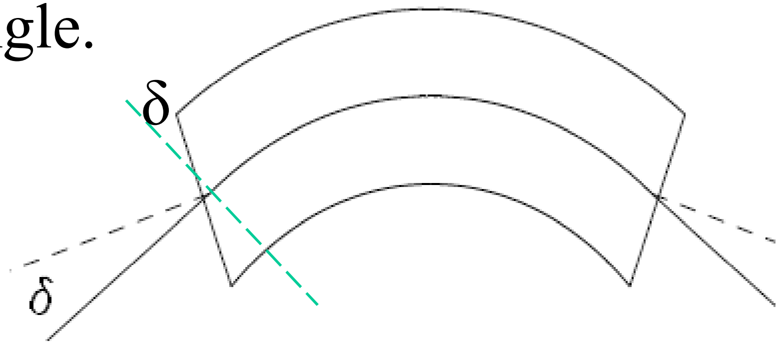
$$M(s,s_0) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

4. Dipole: $K_x(s)=1/\rho^2$. $M(s,s_0)=\begin{pmatrix}\cos\frac{\ell}{\rho} & \rho\sin\frac{\ell}{\rho} \\ -\frac{1}{\rho}\sin\frac{\ell}{\rho} & \cos\frac{\ell}{\rho}\end{pmatrix} \xrightarrow[\frac{\ell}{\rho}\to 0]{} \begin{pmatrix}1 & \ell \\ 0 & 1\end{pmatrix}$

a. When the bend angle is small $\ell/\rho \ll 1$, the transfer matrix of the dipole is equal to that of a drift space.

b. When the bending radius ρ is small, dipoles can provide strong horizontal focusing with $K_x=1/\rho^2$. The vertical focusing is adjusted by trimming the edge angle.

$$M_x = \begin{pmatrix} 1 & 0 \\ \frac{\tan\delta}{\rho} & 1 \end{pmatrix} \quad M_z = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{pmatrix}$$



The particle orbit enters and exits a *sector dipole* magnet perpendicular to the dipole edges. Assuming that the gradient function of the dipole is zero, i.e. $\partial B_z/\partial x = 0$, show that the transfer matrix is

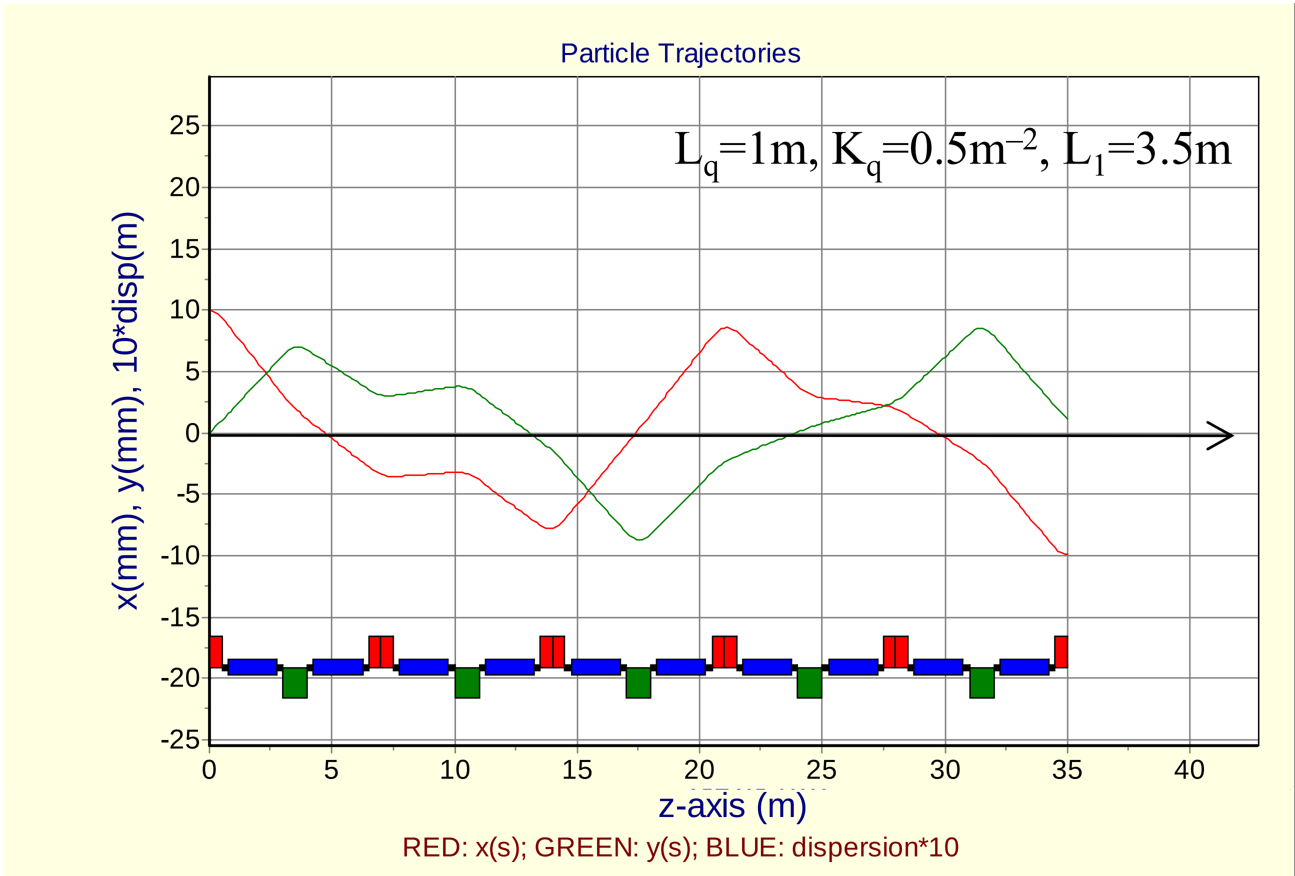
$$M_x = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{\sin\theta}{\rho} & \cos\theta \end{pmatrix}, \quad M_z = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$$

where θ is the bending angle, ρ is the bending radius, and ℓ is the length of the dipole.

Using the transfer matrices, we can express the solution of the Hill's equation: $y'' + K(s)y = 0,$

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = M(s, s_n)M(s_n, s_{n-1})M(s_{n-1}, s_{n-2})\ldots M(s_1, s_0)\begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

$$\begin{pmatrix} x_0 \\ x_0' \\ z_0 \\ z_0' \end{pmatrix} = \begin{pmatrix} 10\text{mm} \\ 0 \\ 0 \\ 2.0\text{mr} \end{pmatrix}$$



Floquet theorem: Many accelerators are designed with the periodic condition: $K(s+L)=K(s)$. The solution of Hill's equation is periodic. In matrix representation, we obtain

$M_1 M_2 \ldots\ldots\ldots\ldots M_n \quad M_1 M_2 \ldots\ldots\ldots\ldots M_n$

$$M(s_1+L|s_1)=M_n M_{n-1} M_{n-2} \ldots M_2 M_1 = \mathbf{M}(s_1)$$

$$M(s_2+L|s_2)=M_1 M_n M_{n-1} M_{n-2} \ldots M_2 = \mathbf{M}(s_2) = M_1 \mathbf{M}(s_1) M_1^{-1}$$

$$M(s_4+L|s_4)=M_3 M_2 M_1 M_n M_{n-1} M_{n-2} \ldots M_4 = \mathbf{M}(s_4)$$

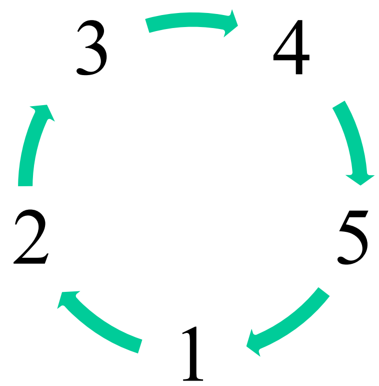
Each **M(s)** matrix is a product of identical number of matrices. They are related by **similarity** transformation. The eigen-values of the periodic matrix $\mathbf{M}(s_i)$ are identical.

The most general representation of the matrix M(s) with **unit modulus** is given by the Courant-Snyder parameterization.

$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix} = I\cos\Phi + J\sin\Phi,$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \quad J^2 = -I, \quad \text{or} \quad \beta\gamma = 1 + \alpha^2$$

$$\lambda = e^{\pm j\Phi} = \cos\Phi \pm j\sin\Phi$$



As particles move through periods of an accelerator, the transfer matrix becomes

$$\mathrm{M}^k = (\mathrm{I}\cos\Phi + \mathrm{J}\sin\Phi)^k = \mathrm{I}\cos k\Phi + \mathrm{J}\sin k\Phi,$$
$$\mathrm{M}^{-1} = \mathrm{I}\cos\Phi - \mathrm{J}\sin\Phi.$$

Example: FODO cell

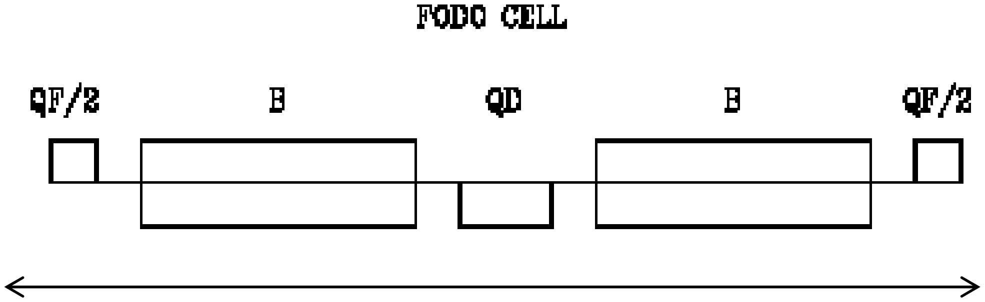
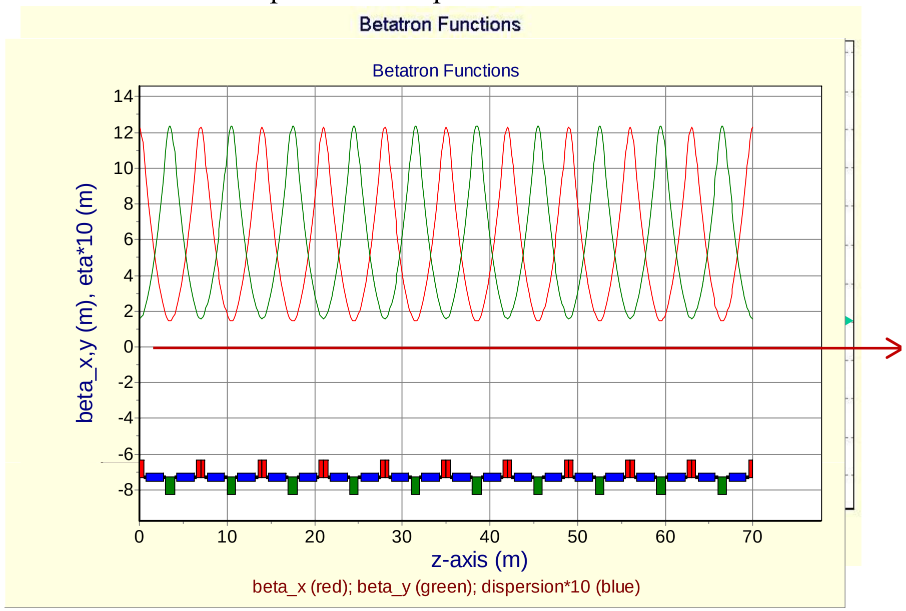


Figure 2.4: A schematic plot of a FODO cell, where the transfer matrix for the dipoles (B) can be approximated by drift spaces, and QF and QD indicate the focusing and defocusing quadrupoles.

$$\mathrm{M} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{L_1^2}{2f^2} & 2L_1(1+\frac{L_1}{2f}) \\ -\frac{L_1}{2f^2}(1-\frac{L_1}{2f}) & 1-\frac{L_1^2}{2f^2} \end{pmatrix}$$

L_q=1m, K_q=0.5m^{-2}, L_1=3.5m



$$M(s) = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\beta = \frac{2L_1(1+\dfrac{L_1}{2f})}{\sin\Phi} = \frac{2L_1(1+\sin\dfrac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

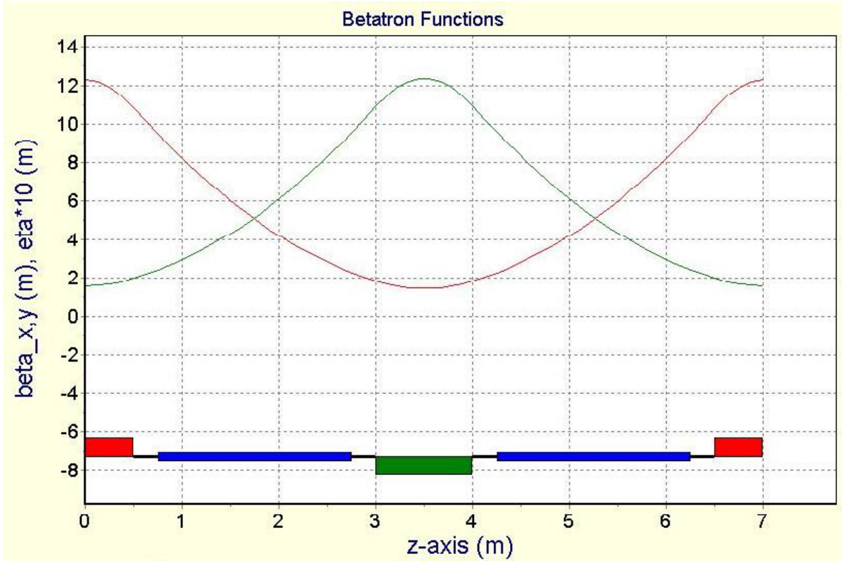
cos(Φ)=Trace(M) $\quad \cos\Phi = 1 - \dfrac{L_1^2}{2f^2}, \quad \sin\dfrac{\Phi}{2} = \dfrac{L_1}{2f}$

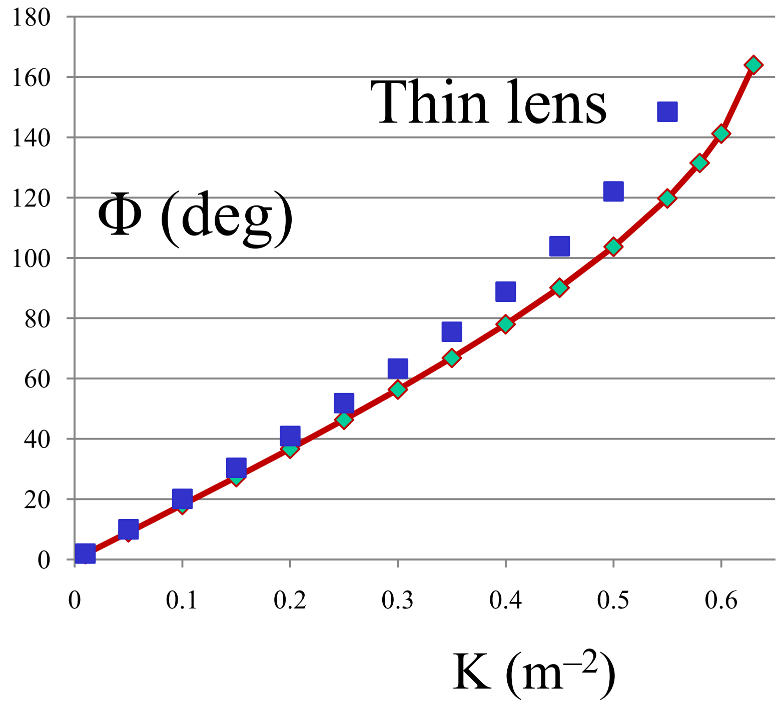
Thin lens – use with care

$$\cos\Phi = 1 - \frac{L_1^2}{2f^2}, \quad \sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

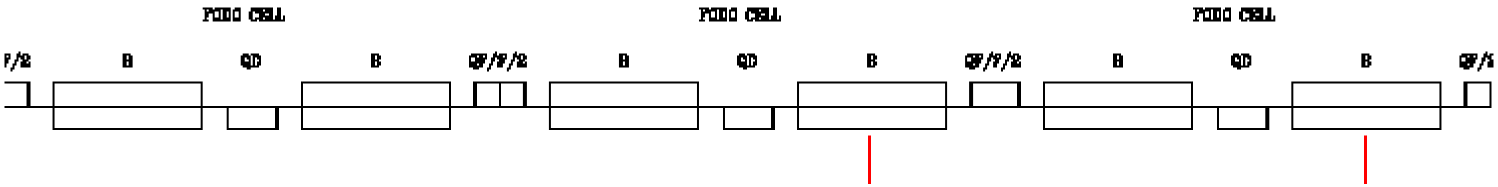
In this FOFDO cell with Lq=1.0 m, L_dipole=2.0 m, drift length of 0.25 m, the thin lens provides us a good approximation. Nevertheless, the percentage error is larger than 11%.

L_q=1m, K_q=1m^{-2}, L_1=3.5m





Example: FODO cell



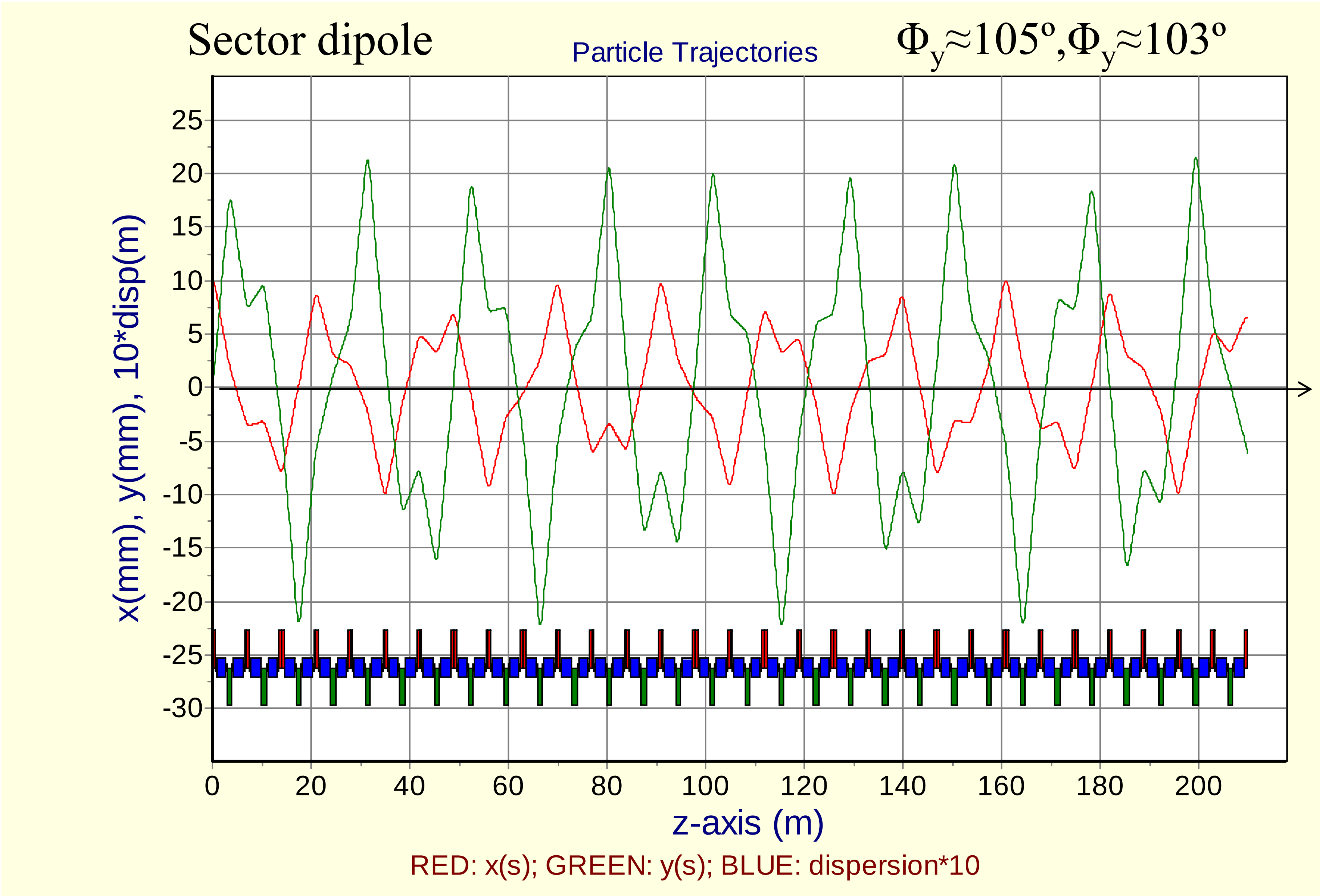
$$\begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{3L}{4} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\begin{aligned} M(s_2) &= I\cos\Phi + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix}\sin\Phi = I\cos\Phi + J_2\sin\Phi \\ M(s_1) &= I\cos\Phi + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix}\sin\Phi = I\cos\Phi + J_1\sin\Phi. \end{aligned}$$

X_0=10 m, x_0'=0, y_0=0, y_0'=5 mrad



$$M(s_2) = M(s_2|s_1)M(s_1)[M(s_2|s_1)]^{-1}.$$

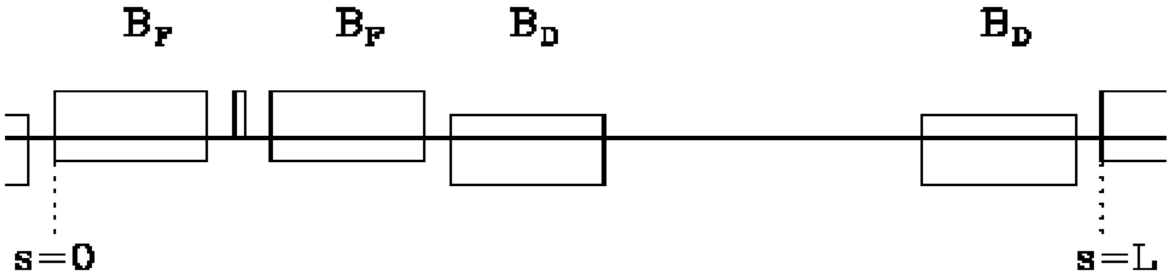
$$\begin{aligned} M(s_2) &= I\cos\Phi + \begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} \sin\Phi = I\cos\Phi + J_2 \sin\Phi \\ M(s_1) &= I\cos\Phi + \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \sin\Phi = I\cos\Phi + J_1 \sin\Phi. \end{aligned}$$

$$J_2 = M(s_2|s_1)J_1[M(s_2|s_1)]^{-1},$$

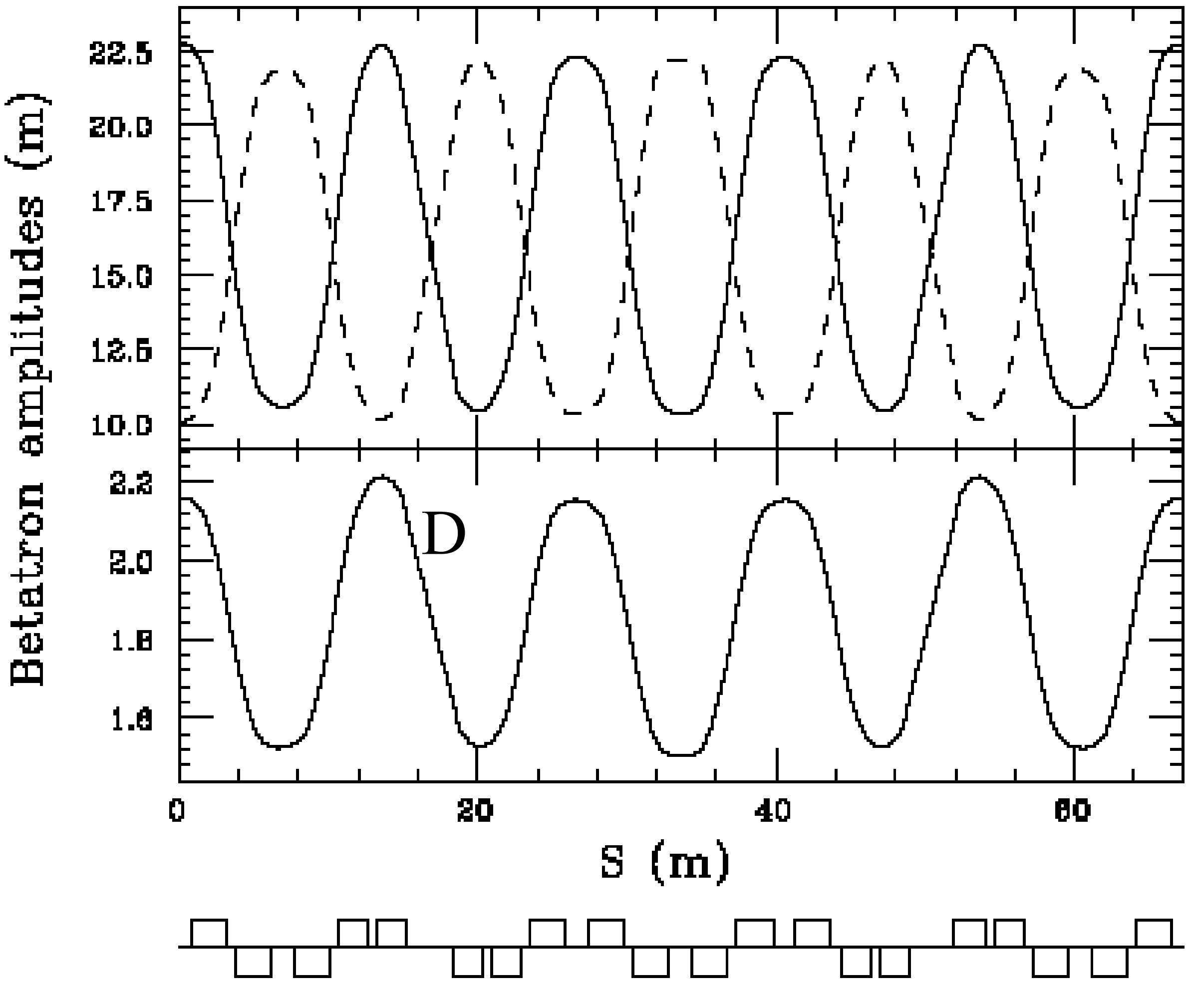
$$\begin{pmatrix} \alpha_2 & \beta_2 \\ -\gamma_2 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 & \beta_1 \\ -\gamma_1 & -\alpha_1 \end{pmatrix} \cdot \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}.$$

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} M_{11}^2 & -2M_{11}M_{12} & M_{12}^2 \\ -M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & -M_{12}M_{22} \\ M_{21}^2 & -2M_{21}M_{22} & M_{22}^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1,$$

M_{ij} is the *ij*-th component of the matrix $M(s_2,s_1)$



Fermilab Booster



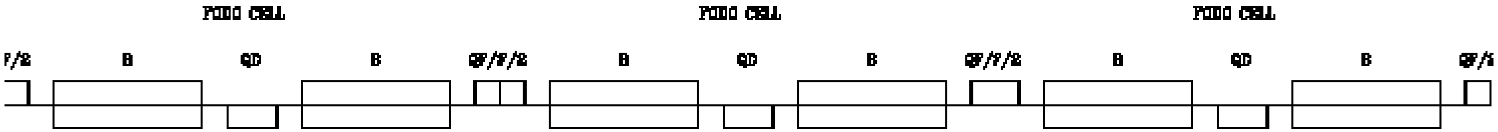
AGS

$$\beta = \frac{2L_1(1+\sin\frac{\Phi}{2})}{\sin\Phi}$$

$$\alpha = 0$$

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f}$$

Stability of accelerator cells: FODO cell example



$$\begin{pmatrix}1 & 0\\ \frac{-1}{f_1} & 1\end{pmatrix}\begin{pmatrix}1 & \mathrm{L}\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{1}{f_2} & 1\end{pmatrix}\begin{pmatrix}1 & \mathrm{L}\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\cos\Phi_x+\alpha_x\sin\Phi_x & \beta_x\sin\Phi_x\\ -\gamma_x\sin\Phi_x & \cos\Phi_x-\alpha_x\sin\Phi_x\end{pmatrix}$$

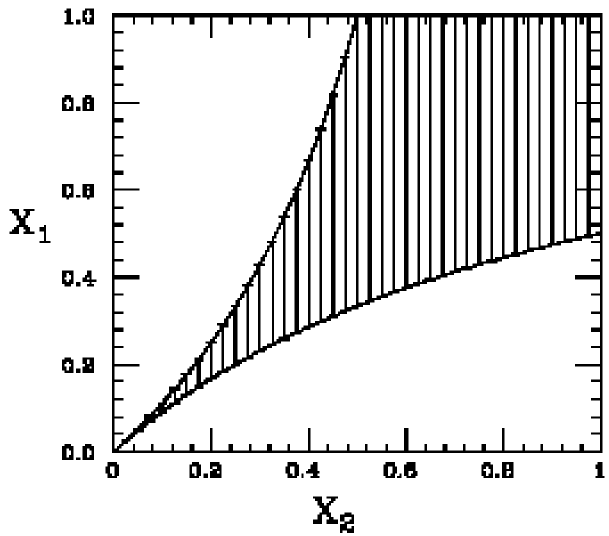
$$\begin{pmatrix}1 & 0\\ \frac{1}{f_1} & 1\end{pmatrix}\begin{pmatrix}1 & \mathrm{L}\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-1}{f_2} & 1\end{pmatrix}\begin{pmatrix}1 & \mathrm{L}\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\cos\Phi_z+\alpha_z\sin\Phi_z & \beta_z\sin\Phi_z\\ -\gamma_z\sin\Phi_z & \cos\Phi_z-\alpha_z\sin\Phi_z\end{pmatrix}$$

$$\cos\Phi_x = 1+\frac{L}{f_2}-\frac{L}{f_1}-\frac{L^2}{2f_1f_2} = 1+2X_2-2X_1-2X_1X_2$$

$$\cos\Phi_z = 1-\frac{L}{f_2}+\frac{L}{f_1}-\frac{L^2}{2f_1f_2} = 1-2X_2+2X_1-2X_1X_2$$

Stability condition: (necktie diagram)

$$|\cos\Phi_x|\le 1,\quad |\cos\Phi_z|\le 1.$$



$$\mathrm{M}(s) = \begin{pmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{n+1} = \mathbf{M}_x \begin{pmatrix} x \\ x' \end{pmatrix}_n , \qquad \begin{pmatrix} z \\ z' \end{pmatrix}_{n+1} = \mathbf{M}_z \begin{pmatrix} z \\ z' \end{pmatrix}_n ,$$

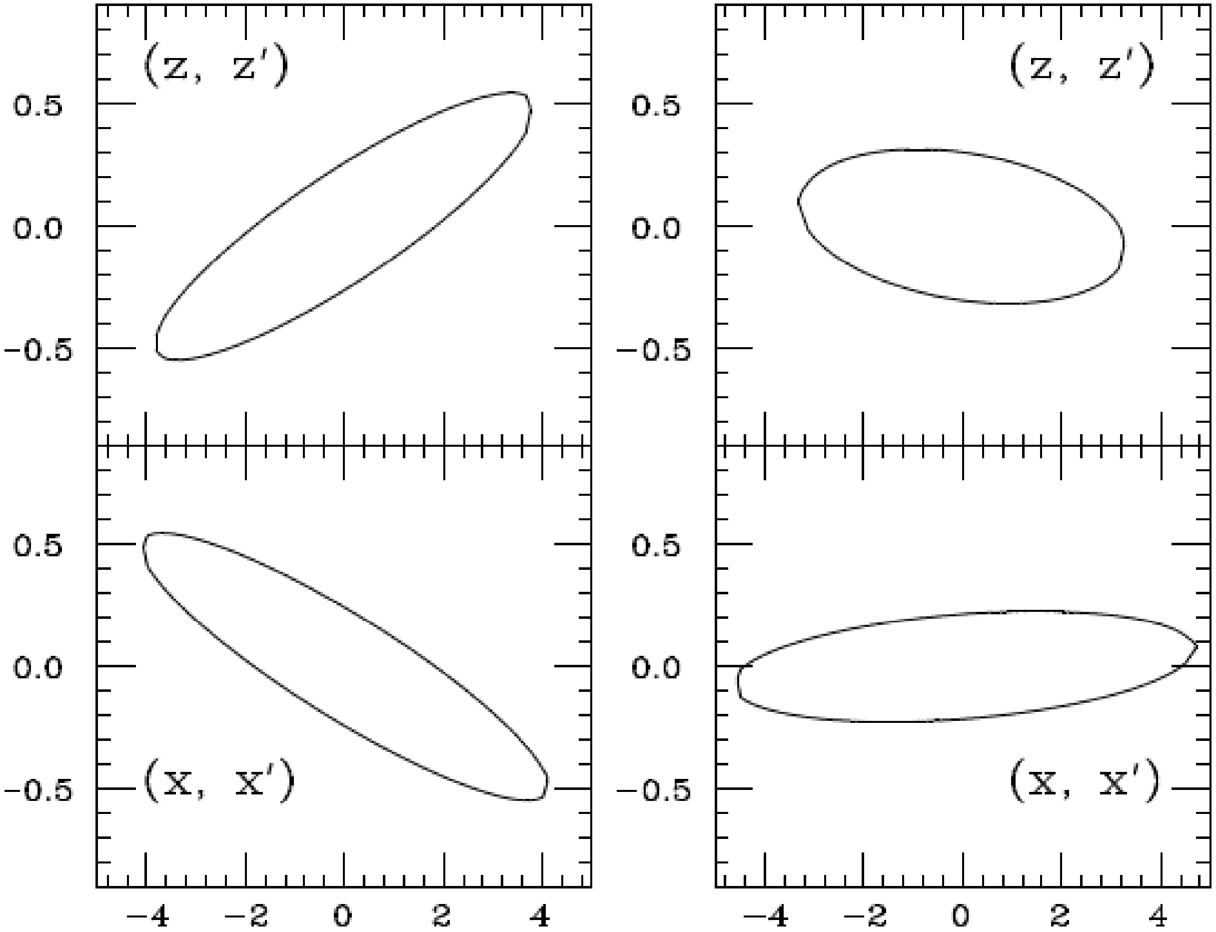


Figure 2.7: The horizontal and vertical betatron ellipses for a particle with actions $J_x = J_z = 0.5\pi$ mm-mrad at the end of the first dipole (left plots) and the end of the fourth dipole of the AGS lattice (see Fig. 2.5). The scale for the ordinate x or z is in mm, and that for the coordinate x' or z' is in mrad. For the left plots, the betatron amplitude functions are $\beta_x = 17.0$ m, $\alpha_x = 2.02$, $\beta_z = 14.7$ m, and $\alpha_z = -1.84$. For the right plots they are $\beta_x = 21.7$ m, $\alpha_x = -0.33$, $\beta_z = 10.9$ m, and $\alpha_z = 0.29$.

Let M be the one-turn map: $$\begin{pmatrix} y \\ y' \end{pmatrix}_n = \mathbf{M}^n \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Let λ_1 and λ_2 be the eigenvalues of the Matrix M and υ_1 and υ_2 be corresponding eigenfunctions. Thus we find:

$$\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = a v_1 + b v_2,$$

$$\begin{pmatrix} y_m \\ y_m' \end{pmatrix} = \mathbf{M}^m \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = a\lambda_1^m v_1 + b\lambda_2^m v_2.$$

The **stability** of particle motion is given by $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$.
This is realized by the condition: $|\mathrm{Trace}(M)| \leq 2$.

Floquet transformation: $y'' + K(s)y = 0,$

$$\boxed{y = aw(s)e^{j\psi(s)}} \qquad w'' + K(s)w - \frac{1}{w^3} = 0 \qquad \psi' = \frac{1}{w^2}$$

Floquet theorem: If K(s)=K(s+L), we can choose the solution with the properties: w(s)=w(s+L), and ψ(s+L)=ψ(s)+Φ, where Φ is the phase advance in one period. The mapping matrix is

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = M(s,s_0)\begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

$$M(s_2,s_1) = \begin{pmatrix} \frac{w_2}{w_1}\cos\psi - w_2 w_1' \sin\psi & w_1 w_2 \sin\psi \\ -\frac{1+w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\psi - (\frac{w_1'}{w_2} - \frac{w_2'}{w_1})\cos\psi & \frac{w_1}{w_2}\cos\psi + w_1 w_2' \sin\psi \end{pmatrix}$$

$$M(s) = \begin{pmatrix} \cos\Phi - ww'\sin\Phi & w^2\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi + ww'\sin\Phi \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos\Phi + \alpha\sin\Phi & \beta\sin\Phi \\ -\gamma\sin\Phi & \cos\Phi - \alpha\sin\Phi \end{pmatrix}$$

$$\beta(s) = w^2, \quad \alpha = -\frac{1}{2}\beta', \quad \gamma = \frac{1+\alpha^2}{\beta}, \qquad w(s) = \sqrt{\beta(s)}, \quad \psi(s) = \int_{s_0}^{s} \frac{1}{\beta}\, ds$$

$$y'' + K(s)y = 0, \qquad H = \frac{1}{2}y'^2 + \frac{1}{2}K(s)y^2,$$

Since $\quad y(s) = a\sqrt{\beta_y(s)}\cos[\psi_y(s) + \xi_y] \quad$ with $\quad \psi_y(s) = \int_0^s \frac{ds}{\beta_y(s)},$

Thus $\quad y' = -\frac{y}{\beta}(\tan\psi - \frac{\beta'}{2}),$

The **generating function** for (y,y') to (ψ,J) transformation is

$$F_1(y, \psi) = \int_0^y y' dy = -\frac{y^2}{2\beta}(\tan\psi - \frac{\beta'}{2}),$$

$$\tilde{H} = H + \frac{\partial F_1}{\partial s} = \frac{J}{\beta}.$$

$$\frac{dJ}{ds} = -\frac{\partial \tilde{H}}{\partial \psi} = 0.$$

$$J = -\frac{\partial F_1}{\partial \psi} = \frac{y^2}{2\beta}\sec^2\psi = \frac{1}{2\beta}[y^2 + (\beta y' + \alpha y)^2].$$

$$\psi' = \partial\tilde{H}/\partial J = 1/\beta(s),$$

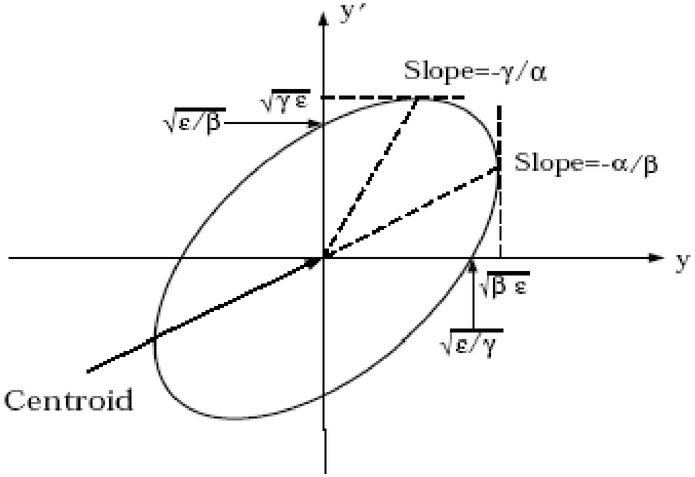
$$y = \sqrt{2\beta J}\ \cos\psi, \qquad y' = -\sqrt{\frac{2J}{\beta}}\ [\sin\psi + \alpha\cos\psi],$$

Define: $\quad \mathcal{P}_y = \beta y' + \alpha y = -\sqrt{2\beta J}\sin\psi.$

(y, P_y) form a normalized phase space coordinates with $y^2 + P_y{}^2 = 2\beta J$, here J is called **action**.

Courant-Snyder Invariant

$$\gamma y^2 + 2\alpha y y' + \beta y'^2 = \frac{1}{\beta}\left[y^2 + (\alpha y + \beta y')^2\right] = 2J \equiv \varepsilon$$



Emittance of a **beam (not a particle)**

$$\langle y\rangle = \int y\rho(y,y')dydy', \quad \langle y'\rangle = \int y'\rho(y,y')dydy',$$

$$\sigma_y^2 = \int (y-\langle y\rangle)^2\rho(y,y')dydy', \quad \sigma_{y'}^2 = \int (y'-\langle y'\rangle)^2\rho(y,y')dydy',$$

$$\sigma_{yy'} = \int (y-\langle y\rangle)(y'-\langle y'\rangle)\rho(y,y')dydy' = r\sigma_y\sigma_{y'},$$

$$\epsilon_{\rm rms} = \sqrt{\sigma_y^2\sigma_{y'}^2 - \sigma_{yy'}^2} = \sigma_y\sigma_{y'}\sqrt{1-r^2}. \qquad \epsilon_{\rm rms} = \sqrt{\det\sigma} = \sqrt{\sigma_{11}\sigma_{22}-\sigma_{12}^2}.$$

A beam with an rms emittance 10 π mm mrad, what will be the rms betatron amplitude of a beam at the location with β=10 m? Ans: 10 mm!

$$\sigma = \begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix} = \begin{pmatrix}\sigma_y^2 & \sigma_{yy'}\\ \sigma_{yy'} & \sigma_{y'}^2\end{pmatrix} = \langle(\mathbf{y}-\langle\mathbf{y}\rangle)(\mathbf{y}-\langle\mathbf{y}\rangle)^\dagger\rangle,$$

$$\sigma(s_2) = M(s_2|s_1)\sigma(s_1)M(s_2|s_1)^\dagger. \qquad \begin{pmatrix}\sigma_x^2 & \sigma_{xx'}\\ \sigma_{xx'} & \sigma_{x'}^2\end{pmatrix} = \epsilon_{\rm rms}\begin{pmatrix}\beta & -\alpha\\ -\alpha & \gamma\end{pmatrix}.$$

$$\rho(y,y') = \rho(\mathbf{y}^\dagger\sigma^{-1}\mathbf{y}).$$

$$\mathbf{x}^\dagger\sigma^{-1}\mathbf{x} = \frac{1}{\epsilon_{\rm rms}}(\gamma x^2 + 2\alpha x x' + \beta x'^2).$$

The rms emittance is invariant in linear transport:

$$\begin{aligned}\epsilon^2 &= \sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2 \\ \sigma_x^2 &= \left\langle x^2\right\rangle - \langle x\rangle^2, \quad \sigma_{x'}^2 = \left\langle x'^2\right\rangle - \langle x'\rangle^2 \quad \sigma_{xx'} = \langle xx'\rangle - \langle x\rangle\langle x'\rangle,\end{aligned}$$

we find

$$\begin{aligned}\frac{d\sigma_x^2}{ds} &= 2\langle xx'\rangle - 2\langle x\rangle\langle x'\rangle \\ \frac{d\sigma_{x'}^2}{ds} &= 2\langle x'x''\rangle - 2\langle x'\rangle\langle x''\rangle \\ \frac{d\sigma_{xx'}}{ds} &= \left\langle x'^2\right\rangle - \langle x'\rangle^2 - \langle x\rangle\langle x''\rangle + \langle xx''\rangle\end{aligned}$$

Using Hamilton's equation $x'' = -\frac{\partial H}{\partial x}$, we find

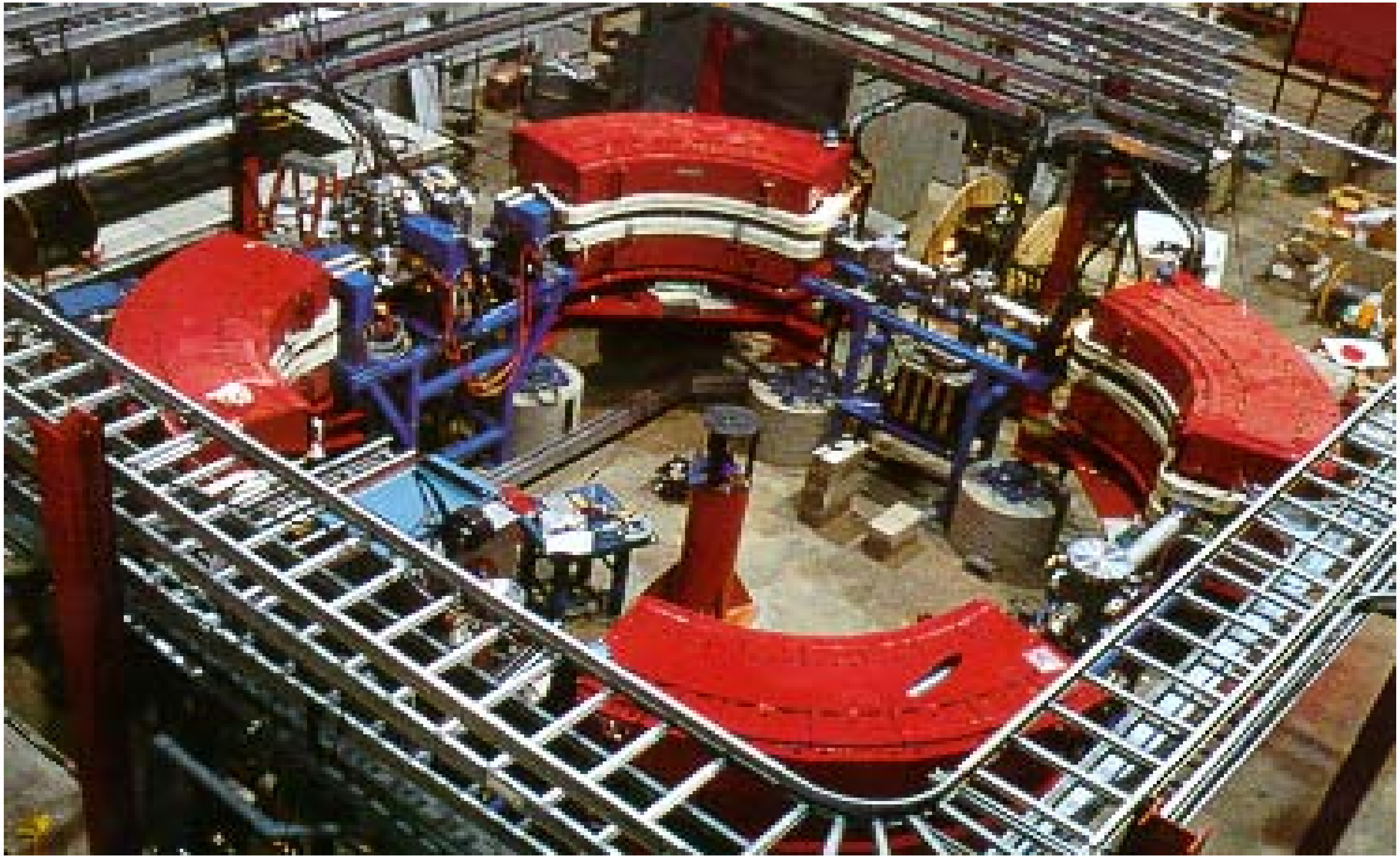
$$\begin{aligned}\frac{d\epsilon^2}{ds} &= \sigma_x^2\frac{d\sigma_{x'}^2}{ds} + \sigma_{x'}^2\frac{d\sigma_x^2}{ds} - 2\sigma_{xx'}\frac{d\sigma_{xx'}}{ds} \\ &= -2\sigma_x^2\left(\left\langle x'\frac{\partial H}{\partial x}\right\rangle - \langle x'\rangle\left\langle\frac{\partial H}{\partial x}\right\rangle\right) + 2\sigma_{xx'}\left(\left\langle x\frac{\partial H}{\partial x}\right\rangle - \langle x\rangle\left\langle\frac{\partial H}{\partial x}\right\rangle\right)\end{aligned}$$

If $\partial H/\partial x = Kx$, then $\frac{d\epsilon^2}{ds} = -2\sigma_x^2\sigma_{xx'} + 2\sigma_{xx'}\sigma_x^2 = 0$, i.e. ϵ is conserved. If the Hamiltonian is nonlinear, ϵ is not invariant.

Some design examples:

1. Large colliders are normally made of arcs and insertion regions (IR), where arcs are made of FODO cells for beam transport, and IRs are used for physics experiments. The IR matches all optical functions for special properties relevant to physics experiments.
2. High power accelerators are designed by taking into account the effects of space charge and transition energy crossing into consideration.
3. Synchrotron radiation facilities are designed to minimize emittance and retain a long straight section for IDs.
4. Low energy proton synchrotrons can use the dipole for horizontal focusing, and edge angle for vertical focusing.

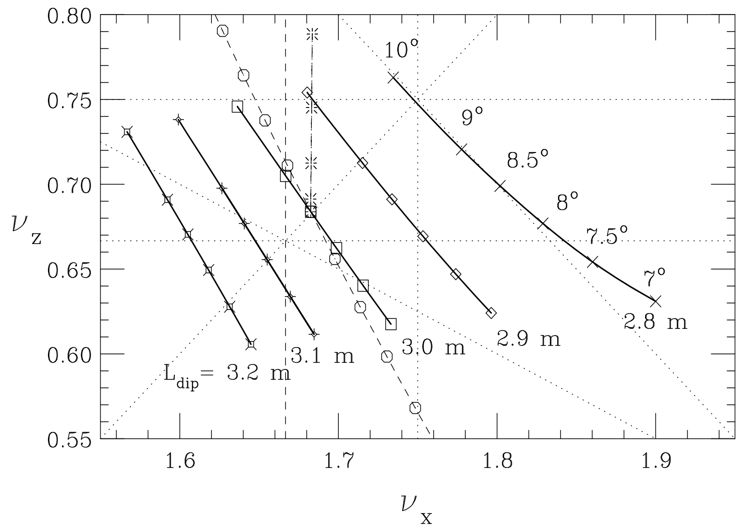
CIS: Circumference =17.364 m, Inj KE= 7 MeV, extraction: 240 MeV
Dipole length = 2 m, 90 degree bend, edge angle = 12 deg.

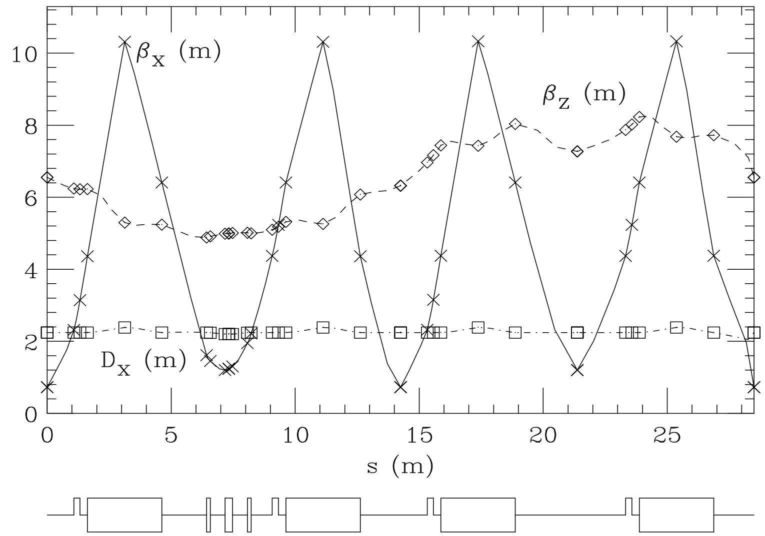


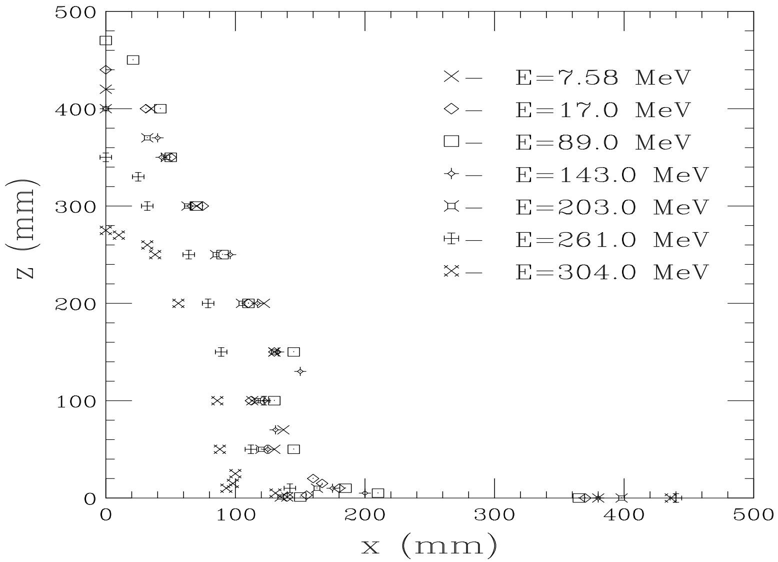
eCIS: No constraint on circumference (C=20m). Use CIS dipoles & cavity
Need Damping wigglers, chicane, electrostatic kickers & septum

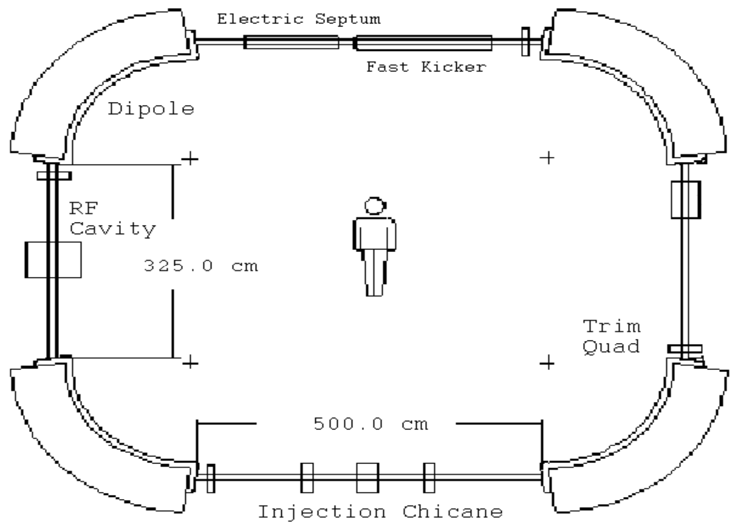
Ldip=3.0 m, ρ=1.91 m, Edge_angle=8.5°

Circum=28.5 m, Qx=1.68, Qz=0.71, KE_tr=356 MeV









$$\sum B\ell = 2\pi B\rho = 2\pi p / e$$

Nader Al Harbi & S.Y. Lee, RSI, **74**, 2540 (2003).

Homework#1

Low energy synchrotrons often rely on the bending radius $K_x = 1/\rho^2$ for horizontal focusing and edge angles in dipoles for vertical focusing. Find the lattice property of the low energy synchrotron described by the following input data file (MAD). What is the effects of changing the edge angle and dipole length? Discuss the stability limit of the lattice.

```
TITLE,"CIS BOOSTER (1/5 Cooler), (90degDIP)"
! CIS =86.82m / 5 =17.364m; protons from 7 MeV to 200 MeV in 1-5 Hz.
LCELL:=4.341     ! cell length 17.364m/4
L1      := 2.0              ! dipole length
L2      :=LCELL-L1   ! straight section length
RHO    :=1.27324
EANG :=12.*TWOPI/360 ! use rad. for edge angle
ANG   := TWOPI/4
OO : DRIFT,L=L2
BD : SBEND,L=L1, ANGLE=ANG, E1=EANG,E2=EANG, K2=0.
SUP: LINE=(BD,OO)     ! a superperiod
USE, SUP, SUPER=4
PRINT, #S/E
TWISS, DELTAP=0.0, TAPE
STOP
```

Betatron motion 2: Effects of Linear Magnetic field Error

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$

$$\Delta B_z + j\Delta B_x = B_0 \sum_n (b_n + ja_n)(x + jz)^n,$$

b_0 : dipole, a_0 : skew (vertical) dipole; $B_z = B_0 b_0, \quad B_x = B_0 a_0,$

b_1 : quad, a_1 : skew quad; $B_z = B_0 b_1 x, \quad B_x = B_0 b_1 z, \quad B_z = -B_0 a_1 z, \quad B_x = B_0 a_1 x,$

Dipole field error: $x'' + K_x(s)x = \dfrac{b_0}{\rho}, \quad z'' + K_z(s)z = -\dfrac{a_0}{\rho}$

Quadrupole field error: $x'' + K_x(s)x = \dfrac{b_1}{\rho}x, \quad z'' + K_z(s)z = -\dfrac{b_1}{\rho}z$

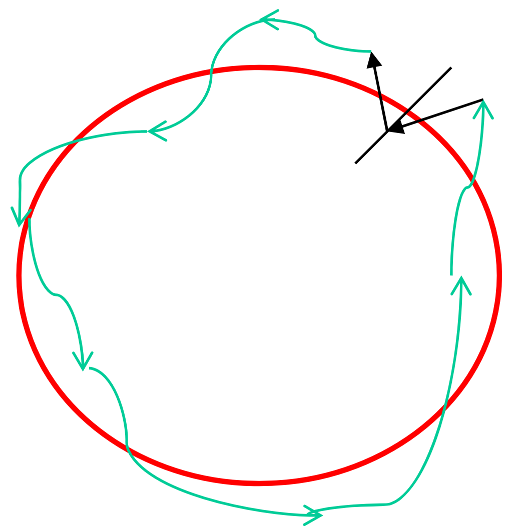
Effect of dipole field error:

We consider a single localized dipole error with the kick angle given by $\theta = \Delta B\ell / B\rho$. Because of the dipole field error, the reference orbit is perturbed! The idea is to find a new closed orbit that include the dipole field error.

$$y'' + K_y(s) y = \theta \delta(s - s_0)$$

The closed orbit condition is:

$$\begin{pmatrix} y_0 \\ y_0' - \theta \end{pmatrix} = M \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$



$$M = \begin{pmatrix} \cos\Phi + \alpha_0 \sin\Phi & \beta_0 \sin\Phi \\ -\gamma_0 \sin\Phi & \cos\Phi - \alpha_0 \sin\Phi \end{pmatrix}$$

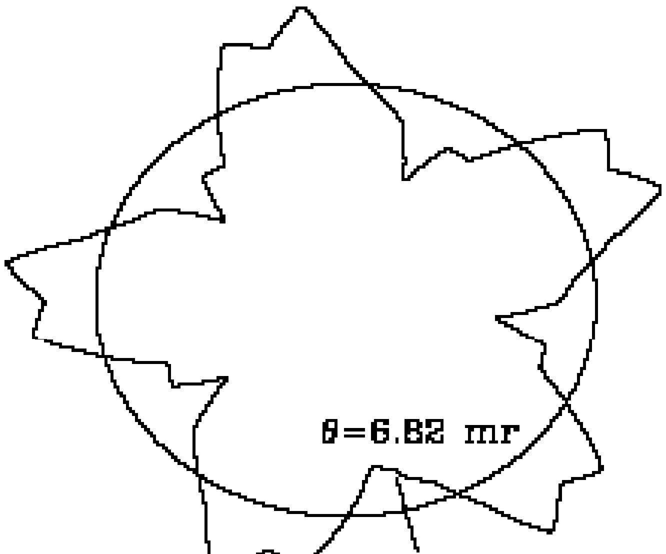
Where $\Phi = 2\pi\nu$, ν is the betatron tune, the parameters α_0, β_0, and γ_0 are values of the Courant-Snyder parameters at the kicker location. The solution is

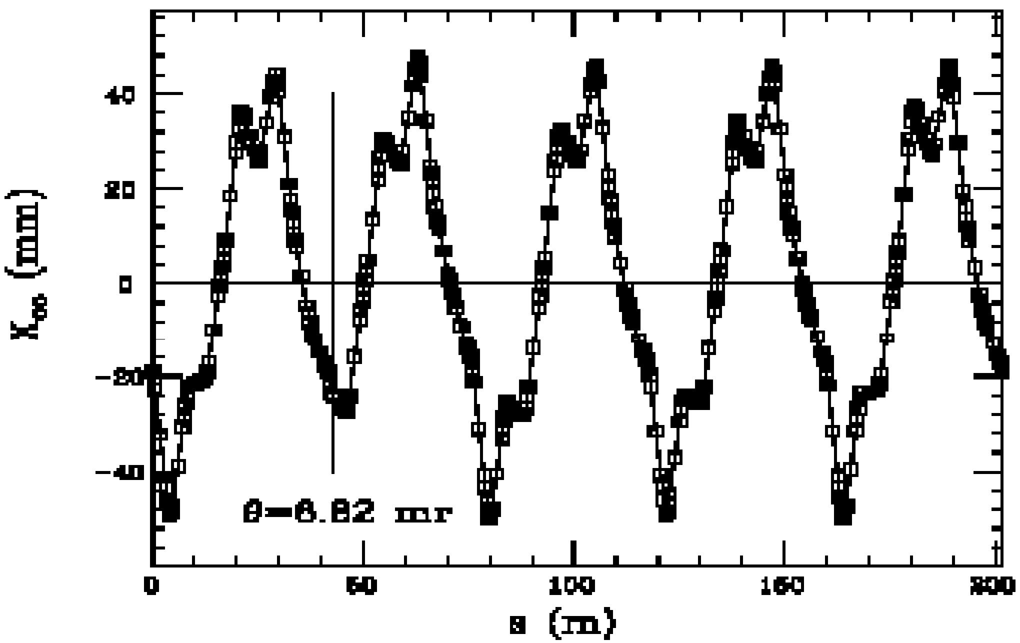
$$y_0 = \frac{\beta_0 \theta}{2 \sin \pi\nu} \cos \pi\nu, \qquad y_0' = \frac{\theta}{2 \sin \pi\nu} (\sin \pi\nu - \alpha_0 \cos \pi\nu)$$

We have solved the closed orbit at one point s_0. The closed orbit of the accelerator can be obtained by making mapping matrix:

$$\begin{pmatrix} y(s) \\ y'(s) \end{pmatrix}_{\text{co}} = M(s, s_0) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \qquad\qquad y_{\text{co}}(s) = G(s, s_0)\theta$$

$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu} \cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$

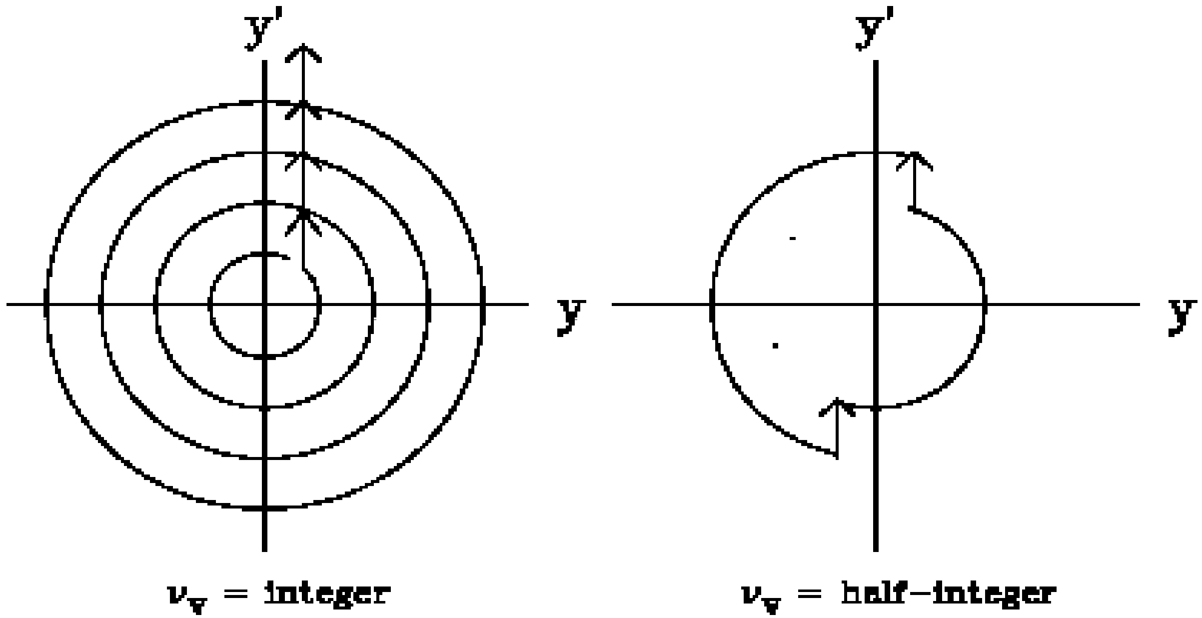




$$y_{\mathrm{co}}(s) = G(s, s_0)\theta$$

$$G(s, s_0) = \frac{\sqrt{\beta(s_0)\beta(s)}}{2\sin\pi\nu}\cos[\pi\nu - |\psi(s) - \psi(s_0)|]$$

Note that the closed orbit is described by Green's function. When the betatron tune is an integer, the closed orbit diverges. Each time, when the particle arrives the same location will receive a coherent kick and the particle becomes unstable.



For the distributed dipole field error, the closed orbit becomes

$$y_{\text{co}}(s)=\frac{\sqrt{\beta(s)}}{2\sin\pi\nu}\int_{s}^{s+C} ds_0\sqrt{\beta(s_0)}\cos[\pi\nu-|\psi(s)-\psi(s_0)|]\frac{\Delta B(s_0)}{B\rho}$$

Making coordinate transformation:

$$\varphi(s)=\frac{1}{\nu}\int_{s_0}^{s}\frac{ds}{\beta(s)},\qquad \psi(s)=\nu\varphi(s),\qquad \psi(s_0)=\nu\varphi(s_0);\quad ds=\nu\beta d\varphi$$

The closed orbit becomes

$$y_{\text{co}}(s)=\frac{\nu\sqrt{\beta(s)}}{2\sin\pi\nu}\int_{s}^{s+C} d\varphi\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]\cos\nu[\pi-|\phi-\varphi|]$$

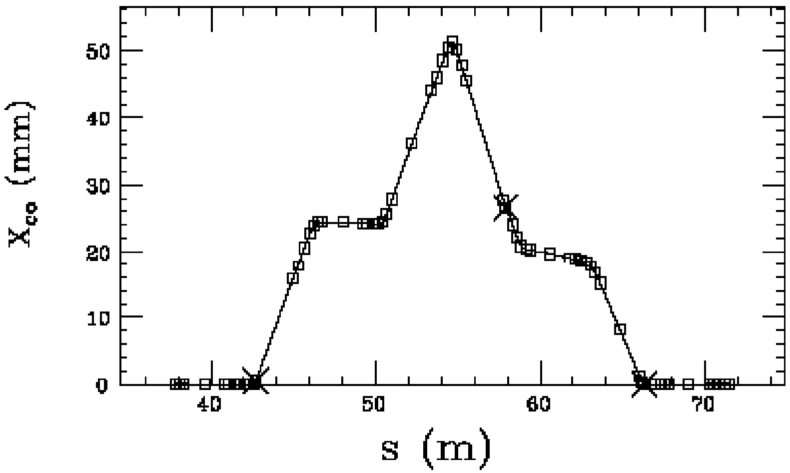
$$\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]=\sum_{k=-\infty}^{\infty} f_k e^{jk\varphi},$$

$$f_k=\frac{1}{2\pi}\oint\left[\beta^{3/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}d\varphi=\frac{1}{2\pi\nu}\oint\left[\beta^{1/2}(\varphi)\frac{\Delta B(\varphi)}{B\rho}\right]e^{-jk\varphi}ds$$

$$y_{\text{co}}(s)=\sqrt{\beta(s)}\sum_{k=-\infty}^{\infty}\frac{\nu^2 f_k}{\nu^2-k^2}e^{jk\phi}$$

Applications of dipole field error:

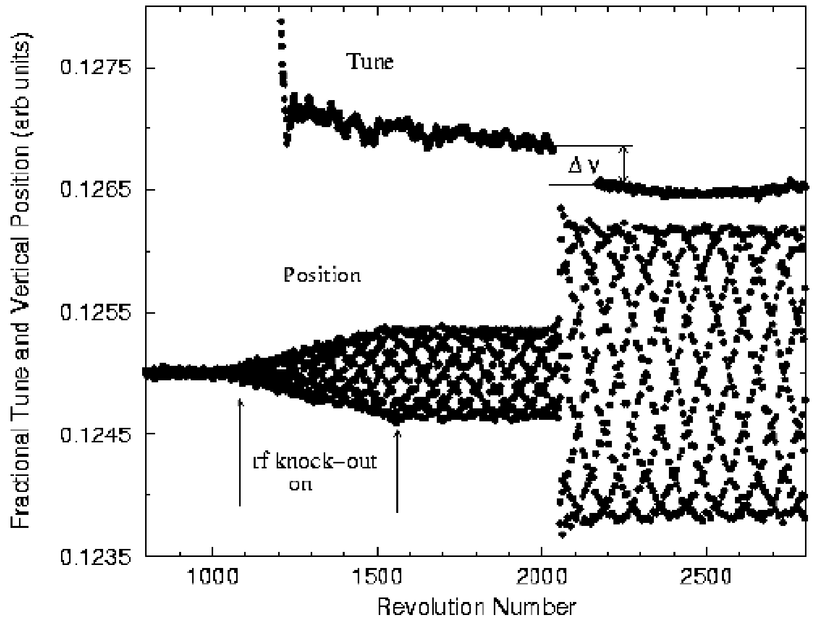
1. closed orbit bump:



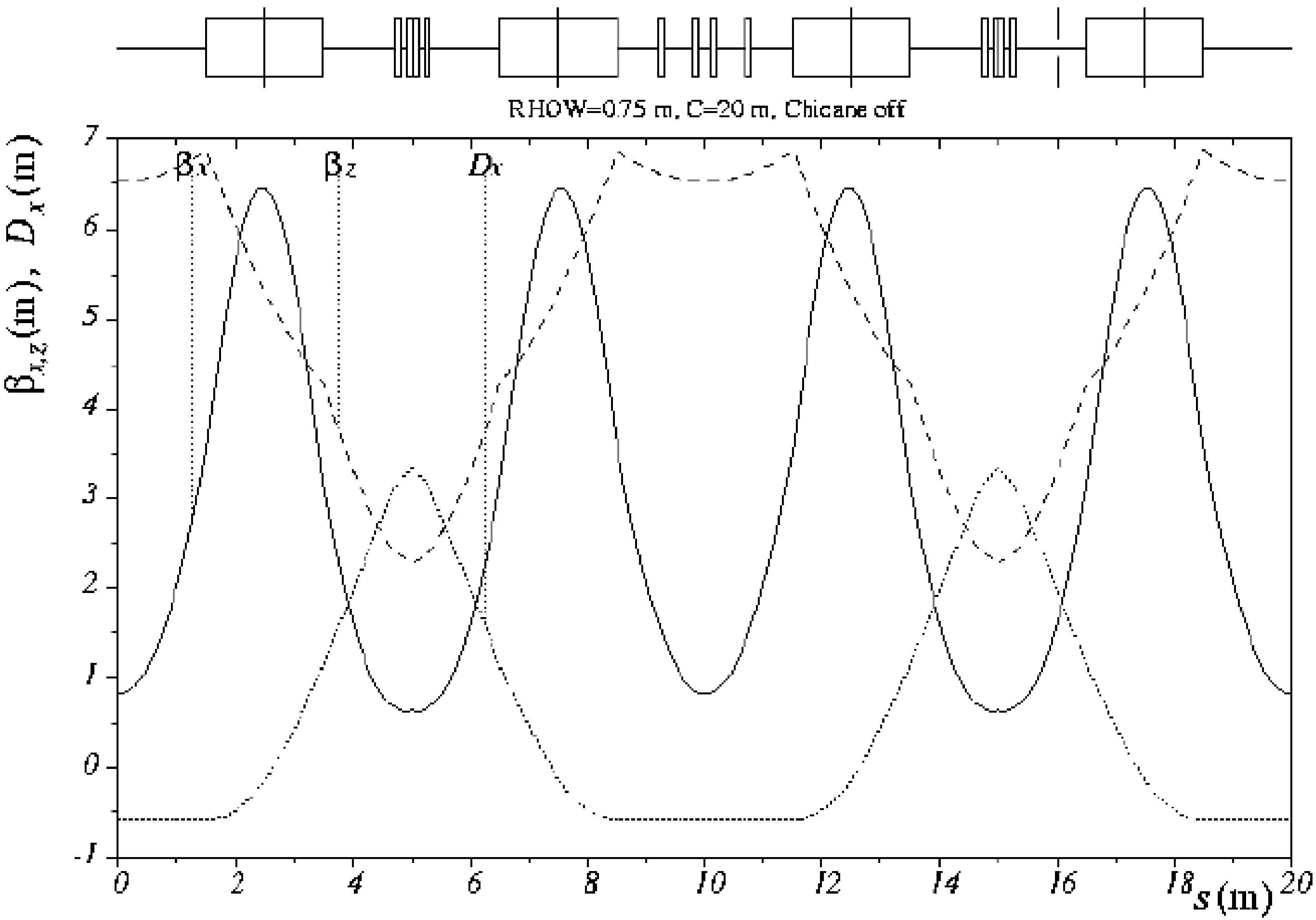
2. injection and extraction kicker

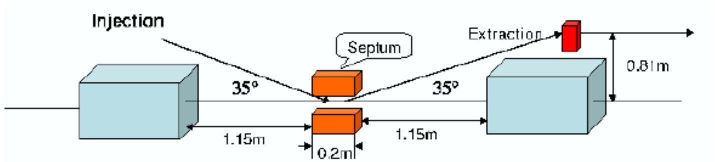
3. rf kicker

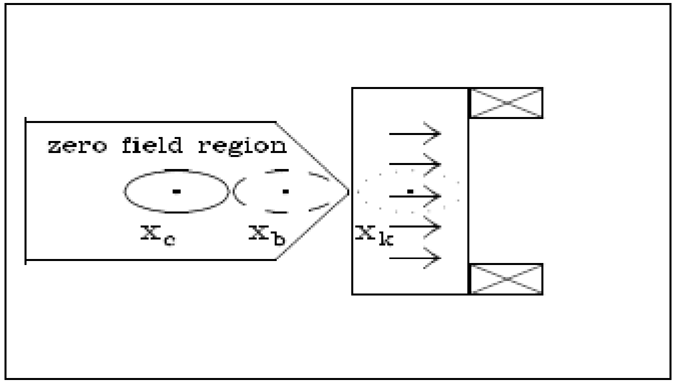
4. …

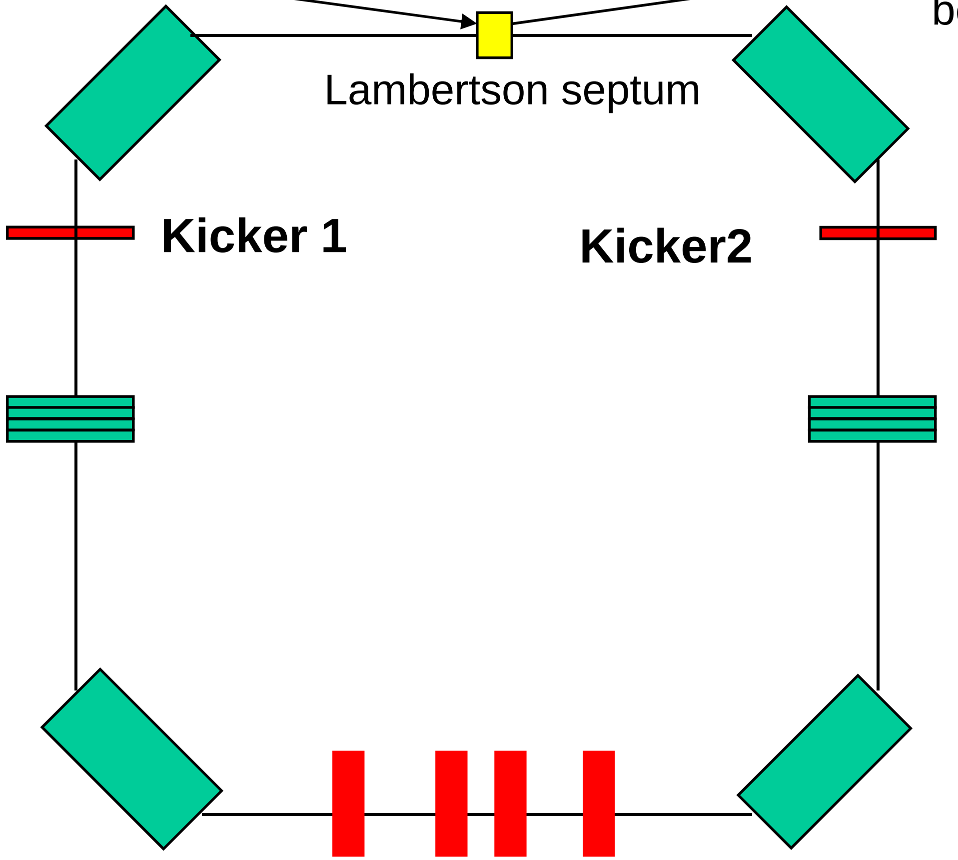


Using chicane-dipoles, one can provide local orbit bumps!









$$x_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi\nu)} \sum_{i=1}^{2} \sqrt{\beta_i}\theta_i \cos\left(\pi\nu - \left|\phi(s) - \phi_i\right|\right)$$

Condition for localized closed orbit :

$$\sqrt{\beta_1}\theta_1 \cos(\pi\nu) + \sqrt{\beta_2}\theta_2 \cos(\pi\nu - \phi_{21}) = 0$$

$$\sqrt{\beta_1}\theta_1 \sin(\pi\nu) + \sqrt{\beta_2}\theta_2 \sin(\pi\nu - \phi_{21}) = 0$$

$$\boxed{\phi_{21} = \pi \quad \text{and} \quad \theta_2 = \theta_1\sqrt{\frac{\beta_1}{\beta_2}}}$$

Kicker Strength

Electrostatic kicker:

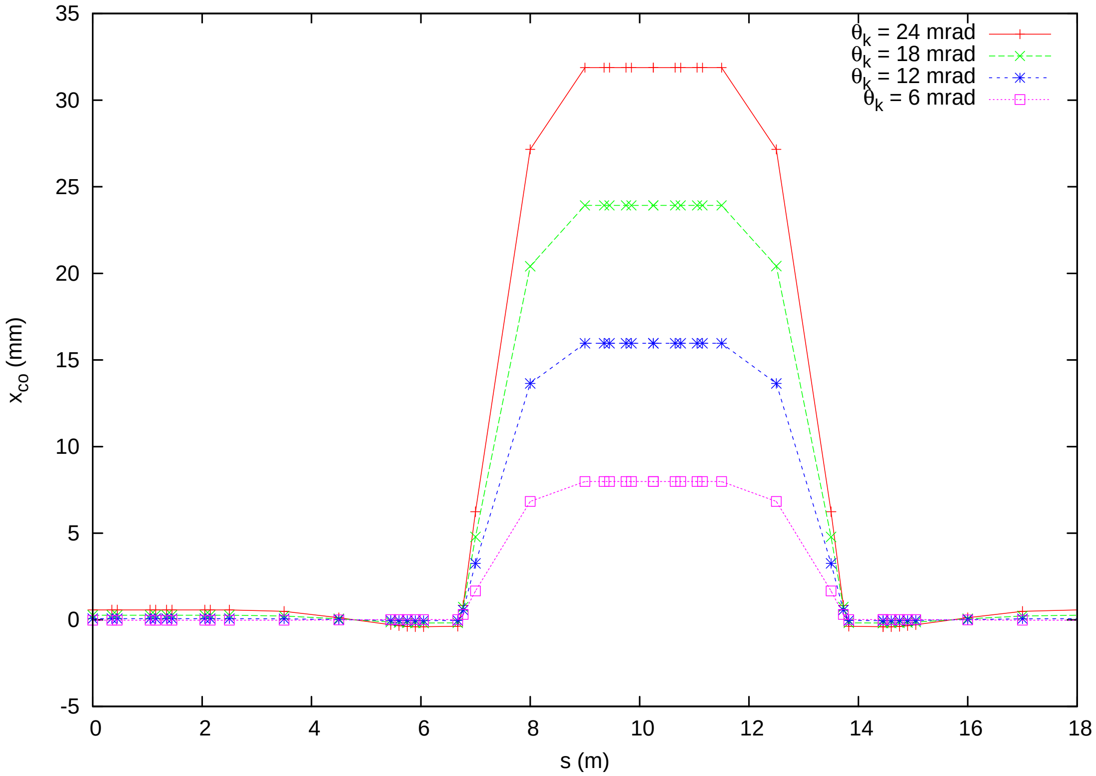
$$\theta_k = \frac{E \cdot L}{c \cdot B\rho} \ , \text{ where}$$

$B\rho = 0.2[Tm]$ at 60 MeV

L = length of the kicker

c = speed of light

E = gap electric field



For one turn injection and extraction, the integrated field strength is 0.60 MV at 25 MeV electron beam energy. Choosing a length of L=0.5 m, the applied voltage on two plate is 60 kV.

Application: Orbit response matrix (ORM) and accelerator modeling

Closed orbit vs dipole field change:

We consider a set of small dipole perturbation given by θ_j, $j = 1, ..., N_{\rm b}$, where $N_{\rm b}$ is the number of dipole kickers. The measured closed orbit from the dipole perturbation is y_i, $i = 1, ..., N_{\rm m}$, where $N_{\rm m}$ is the number of beam position monitors. The response matrix $\mathbf{R}$, defined as

$$y_i = \mathbf{R}_{ij}\theta_j, \quad j = 1, ..., N_{\rm b} \quad i = 1, ..., N_{\rm m},$$

The ORM method minimizes the difference between the measured and model matrices $\mathbf{R}_{\rm exp}$ and $\mathbf{R}_{\rm model}$. Let

$$\mathbf{W}_k = \frac{|\mathbf{R}_{{\rm model},ij} - \mathbf{R}_{{\rm exp},ij}|}{\sigma_i} \qquad \mathbf{R}_{{\rm exp},ij} = \frac{\mathbf{R}_{{\rm data},ij}}{f_j g_i},$$

$$\chi^2 = \frac{1}{N_b \cdot N_m}\sum_k \mathbf{W}_k^2. \qquad ||\mathbf{W}(w_m)|| = 0.$$

$$\mathbf{W}_k(w_m + \Delta w_m) \approx \mathbf{W}_k(w_m) + \frac{d\mathbf{W}_k}{dw_m}\Delta w_m = 0.$$

$$\mathcal{W} = \frac{\mathbf{dW_k}}{\mathbf{dw_m}} = \mathbf{U\Lambda V}^{\rm T}, \qquad \Delta w_m = -\left(\mathbf{V\Lambda}^{-1}\mathbf{U}^{\rm T}\right)\mathbf{W}(w_m),$$

Effect of quadrupole field error:

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho} \quad \rightarrow \; y'' + (K_0 + k)y = 0$$

We assume that the transfer matrix of the unperturbed betatron can be described by

$$M_0(s) = I\cos\Phi_0 + J\sin\Phi_0, \quad \Phi_0 = 2\pi\nu_0 \qquad J(s) = \begin{pmatrix} \alpha(s) & \beta(s) \\ -\gamma(s) & -\alpha(s) \end{pmatrix}$$

The perturbation can be described by $m(s_1) = \begin{pmatrix} 1 & 0 \\ -k(s_1)ds_1 & 1 \end{pmatrix}$

The transfer matrix of the one-turn map is $\mathbf{M}(s_1)=\mathbf{M}_0(s_1)m(s_1)$

$$M(s_1) = \begin{pmatrix} \cos\Phi_0 + \alpha_1\sin\Phi_0 - \beta_1 k(s_1)ds_1 & \beta_1\sin\Phi_0 \\ -\gamma_1\sin\Phi_0 - [\cos\Phi_0 + \alpha_1\sin\Phi_0]k(s_1)ds_1 & \cos\Phi_0 - \alpha_1\sin\Phi_0 \end{pmatrix}$$

$$\cos\Phi = \cos\Phi_0 - \frac{1}{2}\beta_1 k(s_1)ds_1\sin\Phi_0 \qquad \Phi=\Phi_0+\Delta\Phi$$

$$\Delta\Phi \approx \frac{1}{2}\beta_1 k(s_1)ds_1, \quad \Delta\nu \approx \frac{1}{4\pi}\beta_1 k(s_1)ds_1, \quad \Delta\nu \approx \frac{1}{4\pi}\oint\beta_1 k(s_1)ds_1$$

(1) The quadrupole field error changes the betatron tune.

(2) The quadrupole field error also changes the betatron amplitude function, which is obtained by the one-turn map:

$$M(s_2) = M(s_2 + C, s_1) m(s_1) M(s_1, s_2)$$

$$\Delta\left[M(s_2)\right]_{12} = -k(s_1) ds_1 \beta_1 \beta_2 \sin \nu_0(\phi_1 - \phi_2) \sin \nu_0(2\pi + \phi_2 - \phi_1)$$

$$\Delta\left[M(s_2)\right]_{12} = \Delta\left[\beta_2 \sin \Phi\right] \cong \Delta\beta_2 \sin \Phi_0 + \beta_2 \cos \Phi_0 \Delta\Phi$$

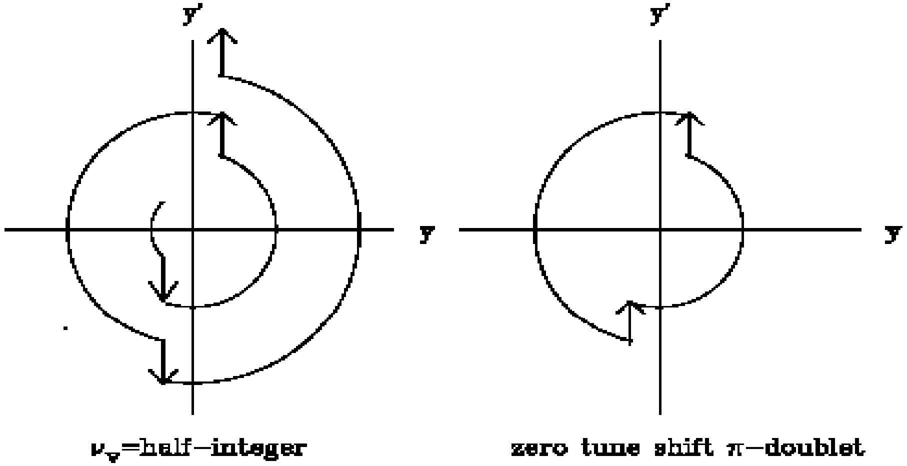
$$\frac{\Delta\beta_2}{\beta_2} = -\frac{1}{2 \sin \Phi_0} \beta_1 k(s_1) ds_1 \sin 2\nu_0(\pi + \phi_2 - \phi_1)$$

For a distributed quadrupole field error, the perturbation to the betatron amplitude function becomes

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{1}{2 \sin \Phi_0} \int_s^{s+C} ds_1 \beta_1 k(s_1) \sin 2\nu_0(\pi + \phi_2 - \phi_1)$$

$$\frac{\Delta\beta(s)}{\beta(s)} = -\frac{\nu_0}{2 \sin \Phi_0} \int_\phi^{\phi+2\pi} d\phi_1 \beta^2(\phi_1) k(\phi_1) \sin 2\nu_0(\pi + \phi - \phi_1)$$

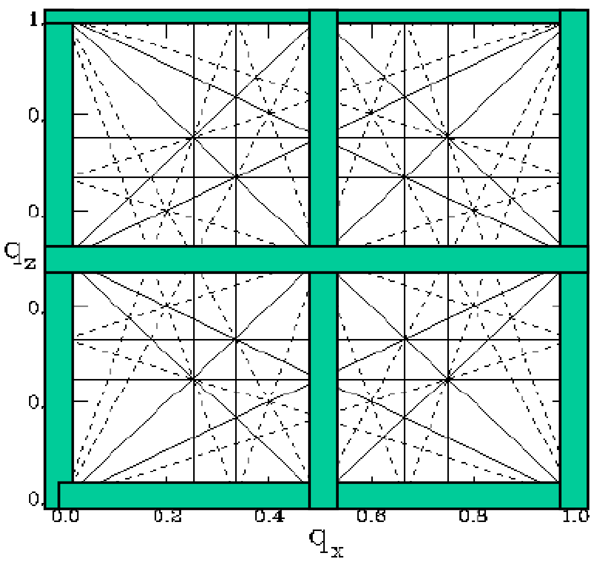
Note that the betatron amplitude function diverges when the betatron tune is integer or half-integer!



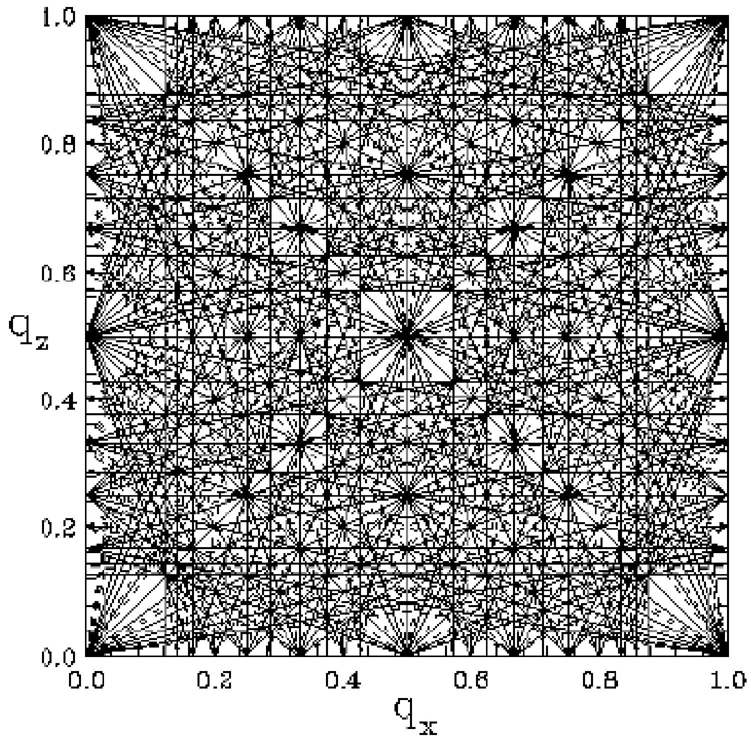
$$\frac{d^2}{d\phi^2}\frac{\Delta\beta(s)}{\beta(s)}+4\nu_0^2\frac{\Delta\beta(s)}{\beta(s)}=-2\nu\beta^2 k(s)$$

$$\left[\nu_0\beta^2 k(s)\right]=\sum_{p=-\infty}^{\infty}J_p e^{jp\varphi},\quad J_p=\frac{1}{2\pi}\oint\left[\beta\, k(s)\right]e^{-jp\varphi}ds$$

$$\frac{\Delta\beta(s)}{\beta(s)}=-\frac{\nu_0}{2}\sum_{p=-\infty}^{\infty}\frac{J_p}{\nu_0^2-(p/2)^2}e^{jp\phi}$$



Effect of Linear resonances



Applications of quadrupole error

Betatron amplitude function measurement

$$\Delta\nu \approx \frac{1}{4\pi}\oint \beta_1 k(s_1) ds_1 \qquad \langle\beta_{x,z}\rangle = 4\pi\frac{\Delta\nu_{x,z}}{\Delta K\ell},$$

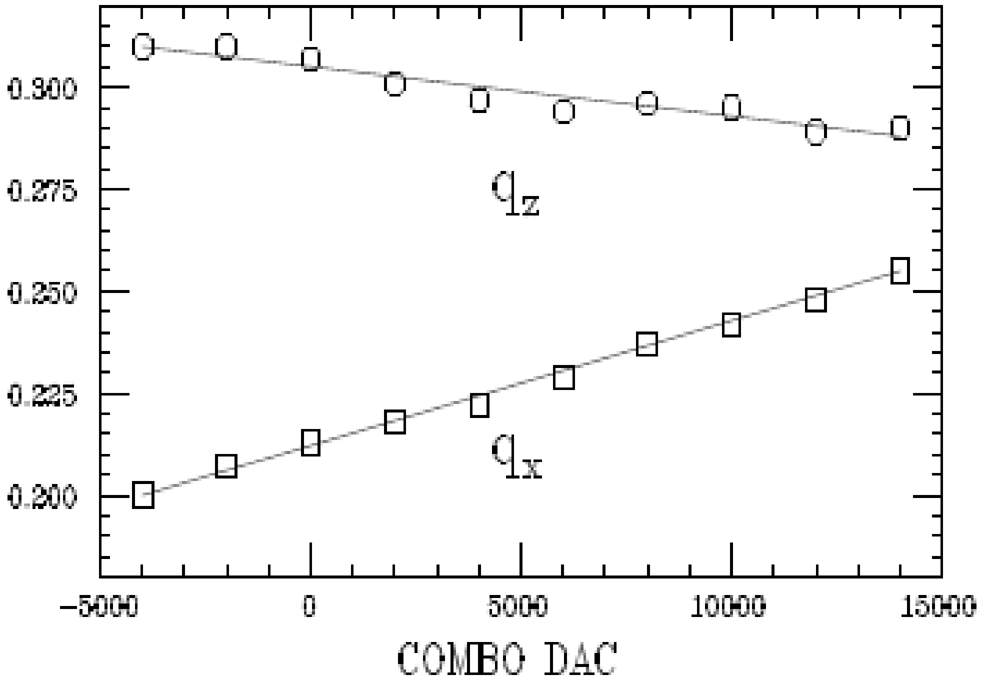


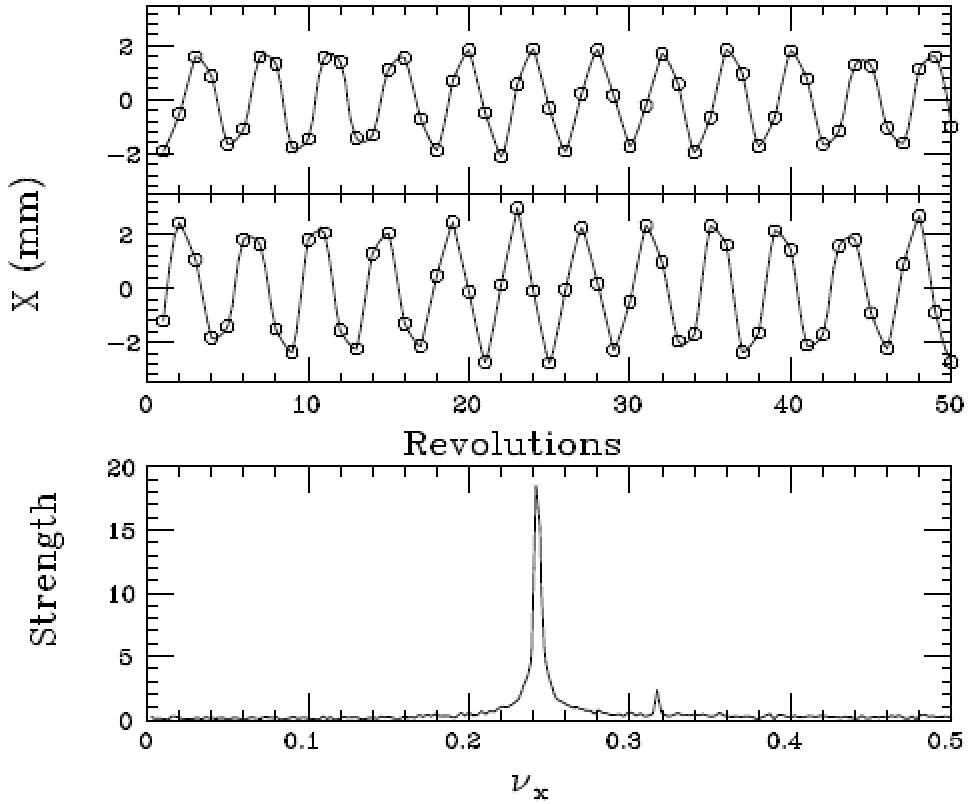
Figure 2.22: An example of betatron amplitude function measurements, where the horizontal and vertical tunes are determined from the FFT spectrum of the betatron oscillations. The slope of the betatron tune vs the quadrupole field variation is used to determine the betatron amplitude functions. Because the fractional parts of betatron tunes are $q_x = 4 - \nu_x$ and $q_z = 5 - \nu_z$, the fractional horizontal tune is seen to increase with the strength of the horizontal defocussing quadrupole.

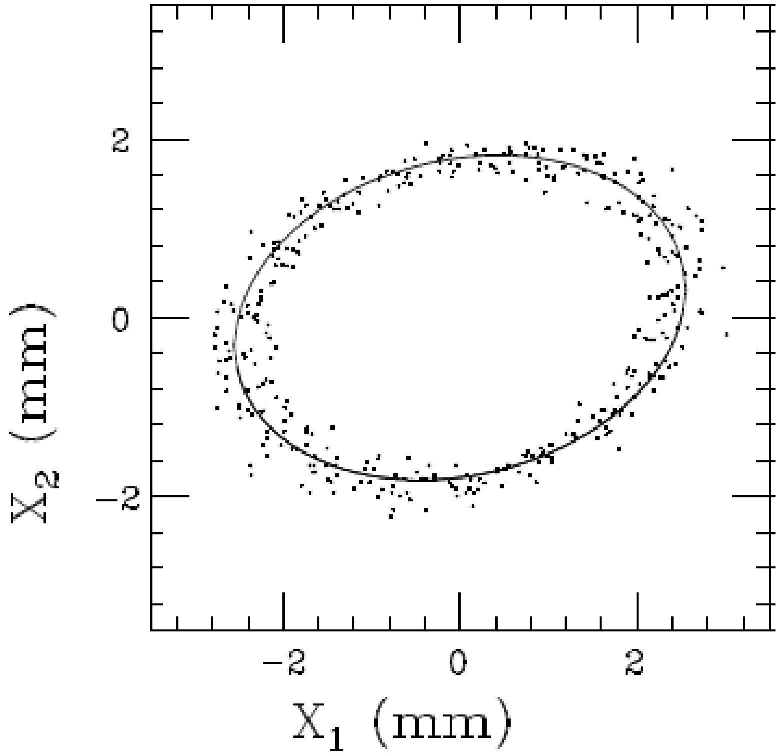
Tune jump

$$\Delta\nu_z = \frac{1}{4\pi}\oint \beta_z \frac{\Delta B_1}{B\rho} ds,$$

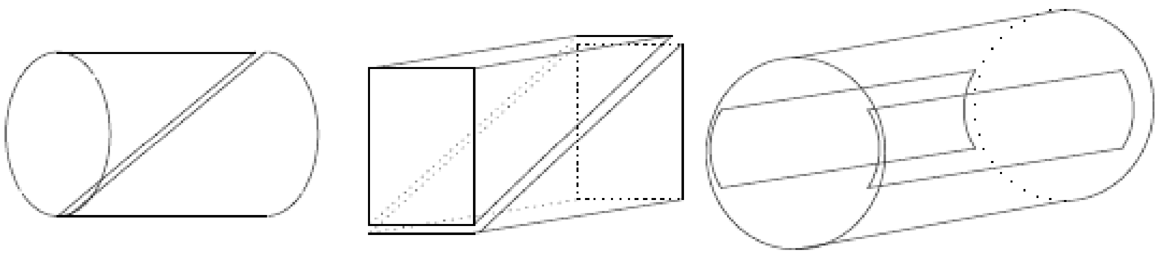
Half-integer stopband corrections

Betatron oscillations in the presence of a dipole kick





$$x_1' = \frac{\csc \psi_{21}}{\sqrt{\beta_1 \beta_2}}\, x_2 - \frac{(\cot \psi_{21} + \alpha_1)}{\beta_1}\, x_1,$$



split can

strip line

Turn-by-turn data can also be used for accelerator modeling

The linear response of a dynamical system is represented by the relation between the N_b-dimensional observation vector $\mathbf{y}(t)$, i.e. the number of BPMs, and the N_s-dimensional source-signal vector $\mathbf{s}(t)$ by

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathcal{N}(t) \tag{B.9}$$

where $N_b \geq N_s$, N_s is unknown a priori, $\mathbf{A} \in \Re^{N_b \times N_s}$ is the mixing matrix, and $\mathcal{N}(t)$ is the noise vector assumed to be stationary, zero mean, temporally white and statistically independent of source signal $\mathbf{s}(t)$. The task is to determine the mixing matrix $\mathbf{A}$

$$\mathbf{y} = \begin{pmatrix} y_1(1) & y_1(2) & \dots & y_1(N) \\ y_2(1) & y_2(2) & \dots & y_2(N) \\ \vdots & \vdots & \ddots & \vdots \\ y_m(1) & y_m(2) & \dots & y_m(N) \end{pmatrix}$$

Consider the betatron motion: $x = \sqrt{2\beta_x J}\,\sin(\nu_x \phi)$

We can organize the turn-by-turn betatron coordinates in a data matrix, *X*. If we carry out singular value decomposition, we find

$$X = U\lambda V^T,$$

$$U = \begin{pmatrix} P\sqrt{\frac{2\beta_x}{M}}\sin(\nu_x\phi_1) & P\sqrt{\frac{2\beta_x}{M}}\cos(\nu_x\phi_1) & \dots \\ P\sqrt{\frac{2\beta_x}{M}}\sin(\nu_x\phi_2) & P\sqrt{\frac{2\beta_x}{M}}\cos(\nu_x\phi_2) & \dots \\ \vdots & \vdots & \dots \end{pmatrix}, \qquad \lambda = \begin{pmatrix} \frac{\sqrt{2JMN}}{2P} & 0 & 0 & \dots \\ 0 & \frac{\sqrt{2JMN}}{2P} & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

$$V^T = \begin{pmatrix} \sqrt{\frac{2}{N}}\cos(2\pi\nu_x\cdot 0) & \sqrt{\frac{2}{N}}\cos(2\pi\nu_x\cdot 1) & \dots & \sqrt{\frac{2}{N}}\cos(2\pi\nu_x\cdot N) \\ \sqrt{\frac{2}{N}}\sin(2\pi\nu_x\cdot 0) & \sqrt{\frac{2}{N}}\sin(2\pi\nu_x\cdot 1) & \dots & \sqrt{\frac{2}{N}}\sin(2\pi\nu_x\cdot N) \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

$$U = (U_1, U_2, \dots, U_n)$$

$$U_1^2 + U_2^2 \propto \beta_x(s)$$

Note that The eigenvalues of betatron modes increase with the number M of BPMs and the number of turns N measured.

What happens if BPM data are noisy? How about the beam energy deviates from the designed value?

$$x = \sqrt{2\beta_x J}\sin(\nu_x\phi) \quad +\mathrm{N}(s,t)+\mathrm{D}(s)\delta(t)$$

Homework#2: Carry out detailed analysis of the betatron motion in the presence of an rf dipole:

$$\frac{d^2y}{ds^2} + K(s)y = \theta_a \sin\omega_{\mathrm{m}} t \sum_{n=-\infty}^{\infty} \delta(s - nC),$$

The key to solve this problem is to carry out Floquet transformation:

$$y'' + K(s)y = \frac{\Delta B}{B\rho} \quad \Longrightarrow \quad \ddot{\eta} + \nu^2\eta = \nu^2\beta^{3/2}\frac{\Delta B}{B\rho}$$

$$\eta = y/\sqrt{\beta} \text{ with } \phi = (1/\nu)\int_0^s ds/\beta$$

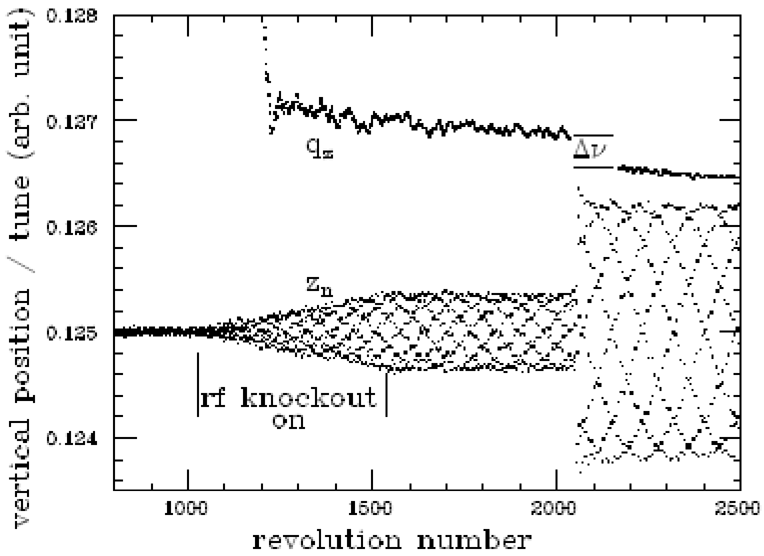
Once you have done this transformation, the rest of the problem become trivial (see next page):

The turn-by-turn data can also be generated by an RF dipole:

$$\frac{d^2y}{ds^2} + K(s)y = \theta_a \sin\omega_{\rm m}t \sum_{n=-\infty}^{\infty} \delta(s-nC), \qquad \eta = \frac{y}{\sqrt{\beta}}, \quad \phi = \frac{1}{\nu}\int_0^s \frac{ds}{\beta},$$

$$\delta(s-nC) = \frac{1}{|ds/d\phi|}\delta(\phi - 2\pi n).$$

$$\frac{d^2\eta}{d\phi^2} + \nu^2\eta = \frac{\nu\sqrt{\beta_0}\theta_a}{2\pi}\sum_{n=-\infty}^{\infty}\sin(n+\nu_{\rm m})\phi,$$



$$\eta = A\cos\nu\phi + B\sin\nu\phi + \eta_{\rm co},$$

$$\eta_{\rm co} = \sum_{n=-\infty}^{\infty}\frac{\nu\sqrt{\beta_0}\theta_a}{2\pi[\nu^2-(n+\nu_{\rm m})^2]}\sin(n+\nu_{\rm m})\phi.$$

$$\begin{aligned} y(s) &= \frac{\sqrt{\beta(s)\beta_0}\ \theta_a}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{\nu^2-(n+\nu_{\rm m})^2}\left[\nu\sin(n+\nu_{\rm m})\phi - (n+\nu_{\rm m})\sin\nu\phi\right] \\ &\approx -\left[\frac{\sqrt{\beta(s)\beta_0}\ \theta_a}{4\pi}\frac{s}{R}\right]\cos\frac{\nu s}{R} + \ldots, \qquad (2. \end{aligned}$$

Off-momentum closed orbit and dispersion function

Including dipole field errors and quadrupole misalignment, we found the closed orbit for a **reference particle with momentum $\mathbf{p_0}$**. By using closed-orbit correctors, we can achieve an optimized closed orbit that essentially passes through the center of all accelerator components. This closed orbit is called the "golden orbit," and a particle with momentum p_0 is called a **synchronous** particle. A beam is made of particles with momenta distributed around the synchronous momentum p_0.

- What happens to particles with momenta different from p_0?
- What is the effect of off-momentum on the closed orbit?

For a particle with momentum p, the momentum deviation is $\Delta p = p - p_0$ and the fractional momentum deviation is $\delta = \Delta p / p_0$, which is typically small of the order of 10^{-6} to 10^{-3}. Since δ is small, we study the motion of off-momentum particles in perturbation expansion of δ.

$$p = p_0 + \Delta p, \quad \delta = \frac{\Delta p}{p_0} << 1$$

$$\begin{cases} x'' - \dfrac{\rho + x}{\rho^2} = \pm \dfrac{B_z}{B\rho} \dfrac{p_0}{p} \left(1 + \dfrac{x}{\rho}\right)^2, \\ z'' = \mp \dfrac{B_x}{B\rho} \dfrac{p_0}{p} \left(1 + \dfrac{x}{\rho}\right)^2, \end{cases}$$

$$B_z = \mp B_0 + B_1 x + \cdots,$$

$$B_0 / B\rho = 1/\rho$$

$$x'' + K_x(s)x = 0, \qquad K_x = 1/\rho^2 \mp K_1(s),$$

$$z'' + K_z(s)z = 0, \qquad K_z = \pm K_1(s),$$

$$p = p_0 + \Delta p, \ \ \delta = \frac{\Delta p}{p_0}$$

$$x'' - \frac{\rho + x}{\rho^2} = \left(-\frac{1}{\rho} + Kx\right) \frac{1}{1+\delta} \left(1 + 2\frac{x}{\rho} + \frac{x^2}{\rho^2}\right)$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K}{(1+\delta)}\right) x = \frac{\delta}{\rho(1+\delta)}$$

$$x'' + \left(\frac{1}{\rho^2} - K(s) + \Delta K(s)\right) x = \frac{\delta}{\rho(1+\delta)}, \quad \Delta K(s) = \left(\frac{2}{\rho^2} - K(s)\right)\delta + O(\delta^2)$$

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_z}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2, \qquad z'' = -\frac{B_x}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2.$$

Expanding the betatron equation of motion, we obtain

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}, \quad K(s) = \frac{B_1}{B\rho}, \quad B_1 = \frac{\partial B_z}{\partial x}$$

For a planar accelerator, the horizontal betatron equation of motion for particles with nonzero δ is inhomogeneous. The solution of the inhomogeneous equation is a linear combination of the particular solution and the solution of the homogeneous equation, i.e.

$$x = x_\beta + D\delta \qquad x' = x'_\beta + D'\delta$$

$$x''_\beta + \left(K_x(s) + \Delta K_x\right)x_\beta = 0, \qquad K_x(s) = \frac{1}{\rho^2} - K(s)$$

$$D'' + \left(K_x(s) + \Delta K_x\right)D = \frac{1}{\rho} + O(\delta)$$

The solution of the homogeneous equation is the betatron oscillation. The solution of the inhomogeneous equation is called the dispersion function, or the off-momentum closed orbit.

$$x'' + \left(\frac{1}{\rho^2} - K(s) + \Delta K(s) \right) x = \frac{\delta}{\rho(1+\delta)}, \quad \Delta K(s) = \left(\frac{2}{\rho^2} - K(s) \right) \delta + O(\delta^2)$$

$$x'' + \left(\frac{1}{\rho^2} - K(s) \right) x = \frac{\delta}{\rho(1+\delta)},$$

$$x = x_\beta + x_{\mathrm{co}} = x_\beta + D\delta$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1) \begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

$$D'' + \left(\frac{1}{\rho^2} - K(s) \right) D = \frac{1}{\rho},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

$$\bar{d} = \begin{cases} \begin{pmatrix} \frac{1}{\rho K_x}(1 - \cos\sqrt{K_x} s) \\ \frac{1}{\rho\sqrt{K_x}} \sin\sqrt{K_x} s \end{pmatrix} & \text{if } K_x > 0, \\ \begin{pmatrix} \frac{1}{\rho |K_x|}(-1 + \cosh\sqrt{|K_x|} s) \\ \frac{1}{\rho\sqrt{|K_x|}} \sinh\sqrt{|K_x|} s \end{pmatrix} & \text{if } K_x < 0. \end{cases}$$

For a pure dipole:

$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -(1/\rho)\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix},$$

$$x = x_\beta + x_{co} = x_\beta + D\delta$$

$$D'' + \left(\frac{1}{\rho^2} - K(s)\right)D = \frac{1}{\rho},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \end{pmatrix} = M(s_2|s_1)\begin{pmatrix} D(s_1) \\ D'(s_1) \end{pmatrix} + \begin{pmatrix} d \\ d' \end{pmatrix},$$

$$\begin{pmatrix} D(s_2) \\ D'(s_2) \\ 1 \end{pmatrix} = \begin{pmatrix} M(s_2|s_1) & \bar{d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix}.$$

For a pure dipole:

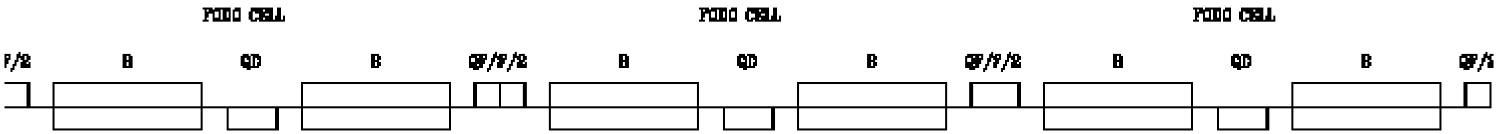
$$M = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{1}{\rho}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \ell & \frac{1}{2}\ell\theta^2 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

For pure quadrupoles:

$$M(s, s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell & 0 \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ -1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M(s, s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell & 0 \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 1/f & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example: FODO cell



$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L & \frac{1}{2}L\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Closed orbit condition:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{L^2}{2f^2} & 2L(1 + \frac{L}{2f}) & 2L\theta(1 + \frac{L}{4f}) \\ -\frac{L}{2f^2} + \frac{L^2}{4f^3} & 1 - \frac{L^2}{2f^2} & 2\theta(1 - \frac{L}{4f} - \frac{L^2}{8f^2}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}.$$

Using the Courant-Snyder parameterization for the transfer matrix, we obtain

$$\sin\frac{\Phi}{2} = \frac{L}{2f}, \quad \beta_{\mathrm{F}} = \frac{2L(1 + \sin(\Phi/2))}{\sin\Phi}, \quad \alpha_{\mathrm{F}} = 0,$$

$$D_{\mathrm{F}} = \frac{L\theta(1 + \frac{1}{2}\sin(\Phi/2))}{\sin^2(\Phi/2)}, \quad D'_{\mathrm{F}} = 0.$$

Result: (1) The dispersion is proportional to the length of the cell L, the bending angle θ, and inversely proportional to the square of the phase advance. (2) The dispersion at other locations can be obtained by using the transfer matrix $M(s_2,s_1)$.

The AGS (33 GeV proton synchrotron built in 1960) is simply made of 60 (5×12) FODO cells. The CPS (28 GeV) is simply made of 50 FODO cells.

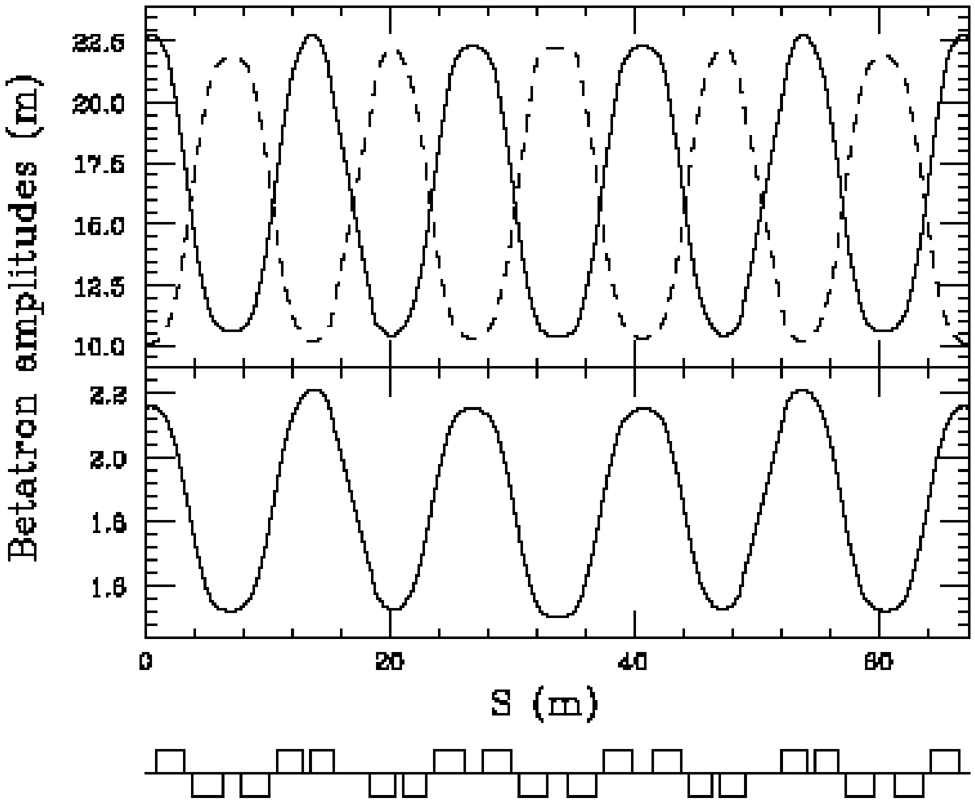


Figure 2.5: The betatron amplitude functions for one superperiod of the AGS lattice, which made of 20 combined-function magnets. The upper plot shows β_x (solid line) and β_z (dashed line). The middle plot shows the dispersion function D_x. The lower plot shows schematically the placement of combined-function magnets. Note that the superperiod can be well approximated by five regular FODO cells. The phase advance of each FODO cell is about 52.8°.

We recall that we define the normalized betatron phase-space coordinates:

$$y^2+P_y{}^2=y^2+(\alpha y+\beta y')^2=2\beta J.$$

We define the normalized dispersion function coordinates:

$$\begin{cases} X_{\rm d} = \dfrac{1}{\sqrt{\beta_x}} D = \sqrt{2J_{\rm d}}\cos\Phi_{\rm d}, \\ P_{\rm d} = \sqrt{\beta_x}D' + \dfrac{\alpha_x}{\sqrt{\beta_x}} D = -\sqrt{2J_{\rm d}}\sin\Phi_{\rm d}, \end{cases}$$

The H-function of the dispersion invariant is defined as:

$$\mathcal{H}(D,D') = \gamma_x D^2 + 2\alpha_x DD' + \beta_x D'^2 = \frac{1}{\beta_x}[D^2 + (\beta_x D' + \alpha_x D)^2].$$

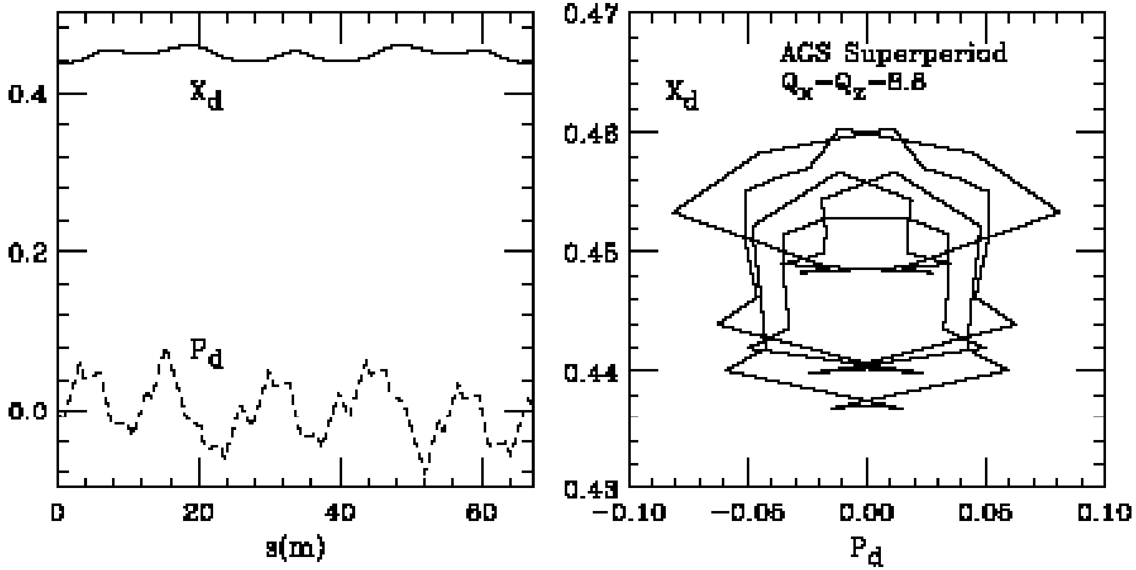


Figure 2.27: Left: Normalized dispersion phase-space coordinates $X_{\rm d}$ and $P_{\rm d}$ are plotted in a superperiod of the AGS lattice. Right: the coordinates are shown in $X_{\rm d}$ vs $P_{\rm d}$. The scales for both $X_{\rm d}$ and $P_{\rm d}$ are $\rm m^{1/2}$.

Path length, momentum compaction and phase-slip factors:

We recall the Frenet-Serret coordinate system. The path length of the reference orbit in one complete revolution is

$$d\ell = \sqrt{(1+\tfrac{x}{\rho})^2(ds)^2+(dx)^2+(dz)^2} = ds[(1+\tfrac{x}{\rho})^2+x'^2+z'^2]^{1/2} \cong ds[1+\tfrac{x}{\rho}]$$

$$C = \oint d\ell \cong \oint ds + \oint \frac{x}{\rho}ds = C_0 + \oint \frac{x_\beta + D\delta}{\rho}ds$$

$$\Delta C = \oint \frac{D}{\rho}ds\,\delta, \qquad \alpha_{\mathrm{c}} \equiv \frac{d\Delta C}{Cd\delta} = \frac{1}{C}\oint \frac{D}{\rho}ds \cong \frac{1}{C}\sum_i \langle D_i\rangle \theta_i$$

Here α_c is called the momentum compaction factor, which is a measure of the compactness of the orbit length for particles with different momenta. The important of the orbit length is that the particles in synchrotron must synchronize with the rf accelerating voltage. Note that the orbiting time for particle is T=C/v. Thus

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_{\mathrm{c}} - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta, \qquad \eta = \alpha_{\mathrm{c}} - \frac{1}{\gamma^2} = \frac{1}{\gamma_{\mathrm{T}}^2} - \frac{1}{\gamma^2}.$$

Here η is called the phase slip-factor.

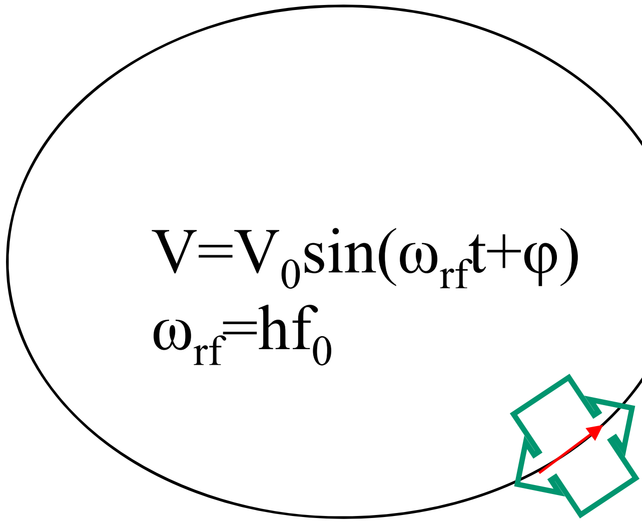
$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta, \qquad \eta = \alpha_c - \frac{1}{\gamma^2} = \frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}.$$

$$V_s = V_0 \sin\phi_s, \quad \dot{E}_0 = f_0 e V_0 \sin\phi_s, \quad \dot{E} = f e V_0 \sin\phi,$$

$\phi = -h\theta$, where θ is the actual angular position of the particle

$$\frac{d}{dt}(\phi - \phi_s) = -h\Delta\omega = h\omega_0 \frac{\Delta T}{T_0} = h\eta\omega_0 \frac{\Delta p}{p_0} = \frac{\eta h \omega_0^2}{\beta^2 E_0}\frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi} e V_0 (\sin\phi - \sin\phi_s),$$



$$\frac{\Delta \upsilon}{\upsilon} = \frac{1}{\gamma^2}\frac{\Delta p}{p_0} = \frac{1}{\beta^2\gamma^2}\frac{\Delta E}{E_0}$$

$$\frac{\dot{E}}{\omega} - \frac{\dot{E}_0}{\omega_0} = \frac{\dot{E} - \dot{E}_0}{\omega_0} - (\frac{1}{\omega_0} - \frac{1}{\omega})\dot{E} \approx \frac{\dot{E} - \dot{E}_0}{\omega_0} + \dot{E}\frac{\Delta(1/\omega_0)}{\Delta E}\Delta E = \frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right)$$

Phase stability and the synchrotron equation of motion:

$$\frac{d}{dt}(\phi - \phi_{\rm s}) = -h\Delta\omega = h\omega_0\frac{\Delta T}{T_0} = h\eta\omega_0\frac{\Delta p}{p_0} = \frac{\eta h\omega_0^2}{\beta^2 E_0}\frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi}eV_0(\sin\phi - \sin\phi_{\rm s}),$$

$$\frac{d^2(\phi - \phi_{\rm s})}{dt^2} = \frac{\eta h\omega_0^2 eV_0}{2\pi\beta^2 E_0}(\sin\phi - \sin\phi_{\rm s}) \approx \frac{\eta\cos\phi_{\rm s} h\omega_0^2 eV_0}{2\pi\beta^2 E_0}(\phi - \phi_{\rm s}).$$

$$\eta\cos\phi_{\rm s} < 0,$$

$$\begin{cases} 0 \le \phi_{\rm s} \le \pi/2 & \text{if } \gamma < \gamma_{\rm T} \text{ or } \eta < 0, \\ \pi/2 \le \phi_{\rm s} \le \pi & \text{if } \gamma > \gamma_{\rm T} \text{ or } \eta > 0. \end{cases}$$

$$\eta = \alpha_{\rm c} - \frac{1}{\gamma^2} = \frac{1}{\gamma_{\rm T}^2} - \frac{1}{\gamma^2}.$$

$$\omega_{\rm syn} = \omega_0\sqrt{\frac{heV_0|\eta\cos\phi_{\rm s}|}{2\pi\beta^2 E_0}}.$$

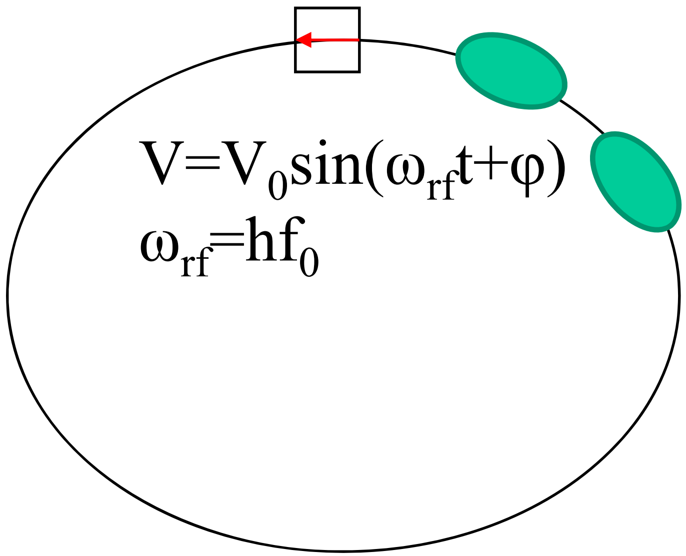
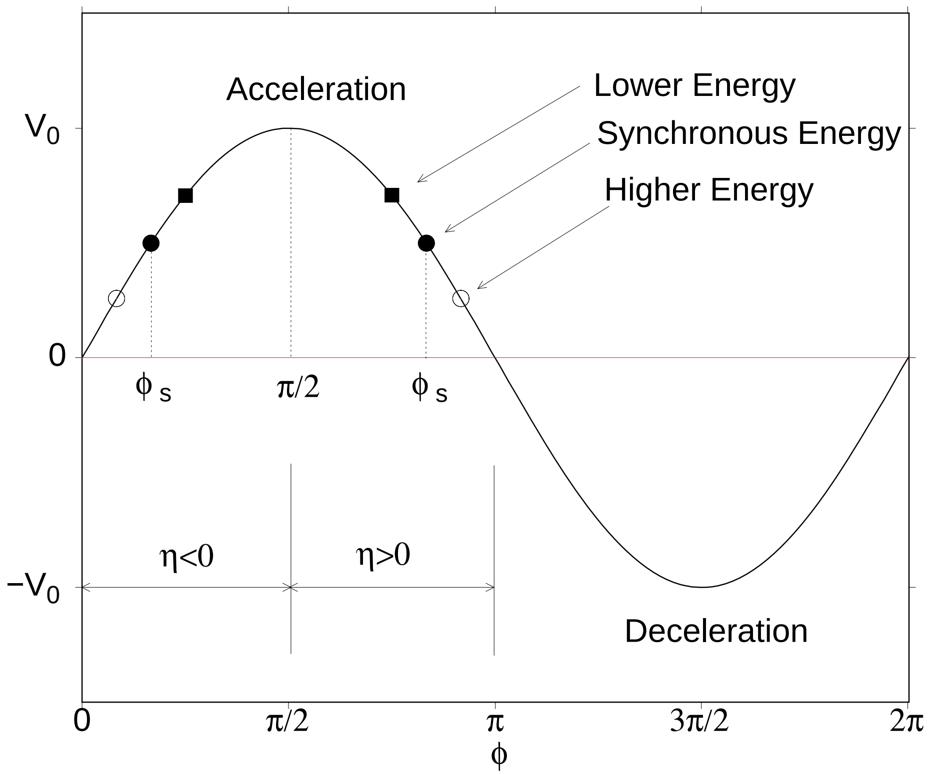


Illustration of the Phase stability: A beam bunch consists of particles with slightly different momenta. A particle with momentum p has its own off-momentum closed orbit $D\delta$. Since the energy gain depends sensitively on the synchronization of rf field and particle arrival time, what happens to a particle with a slightly different momentum when the synchronous particle is accelerated?



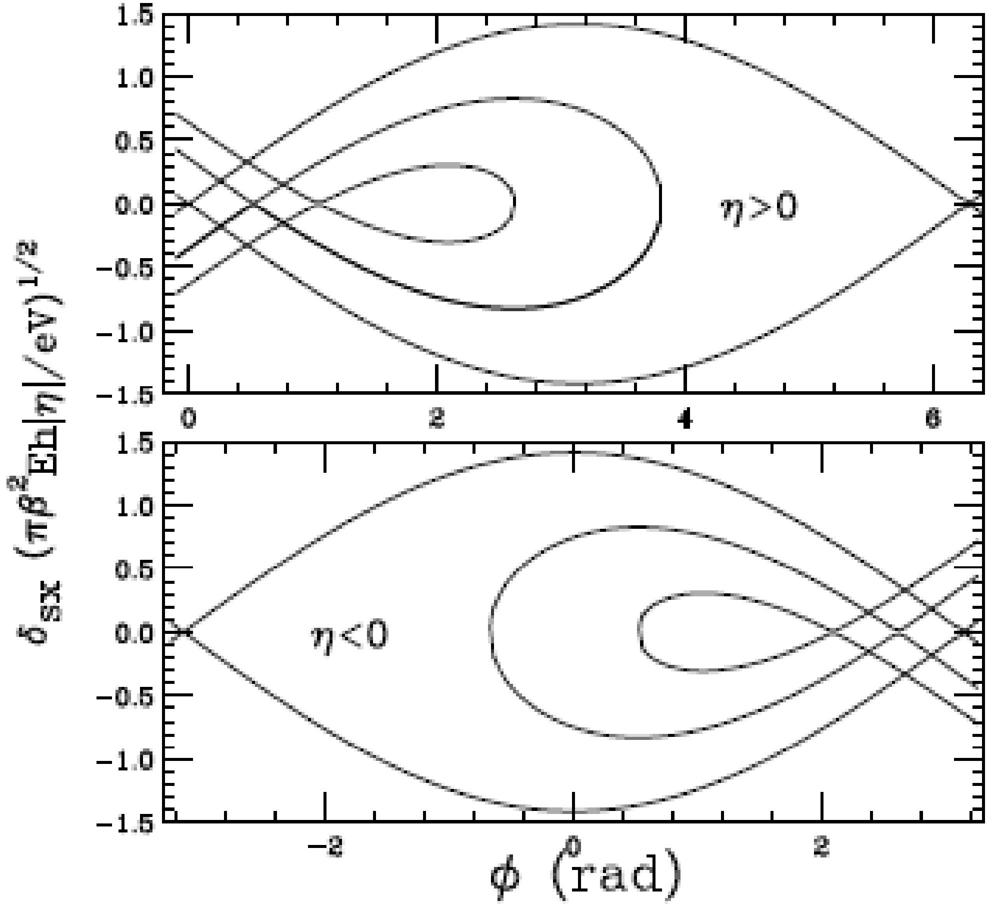
The key answer is the discovery of the phase stability of synchrotron motion by McMillan and Veksler. If the revolution frequency f is higher for a higher momentum particle, i.e. $df/d\delta>0$, the higher energy particle will arrive at the rf gap earlier, i.e. $\varphi<\varphi_s$. Therefore if the rf wave synchronous phase is chosen such that $0<\varphi_s<\pi/2$, higher energy particles will receive less energy gain from the rf gap.

Similarly, lower energy particles will arrive at the same rf gap later and gain more energy than the synchronous particle. This process provides the phase stability of synchrotron motion. In the case of $df/d\delta<0$, phase stability requires $\pi/2<\varphi_s<\pi$.

Synchrotron equation of motion:

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s) \qquad \phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2 E}\Delta E_{n+1}$$

$$\omega_s = \omega_0 \sqrt{\frac{heV|\eta_0 \cos\phi_s|}{2\pi\beta^2 E}} = \frac{c}{R}\sqrt{\frac{heV|\eta\cos\phi_s|}{2\pi E}},$$



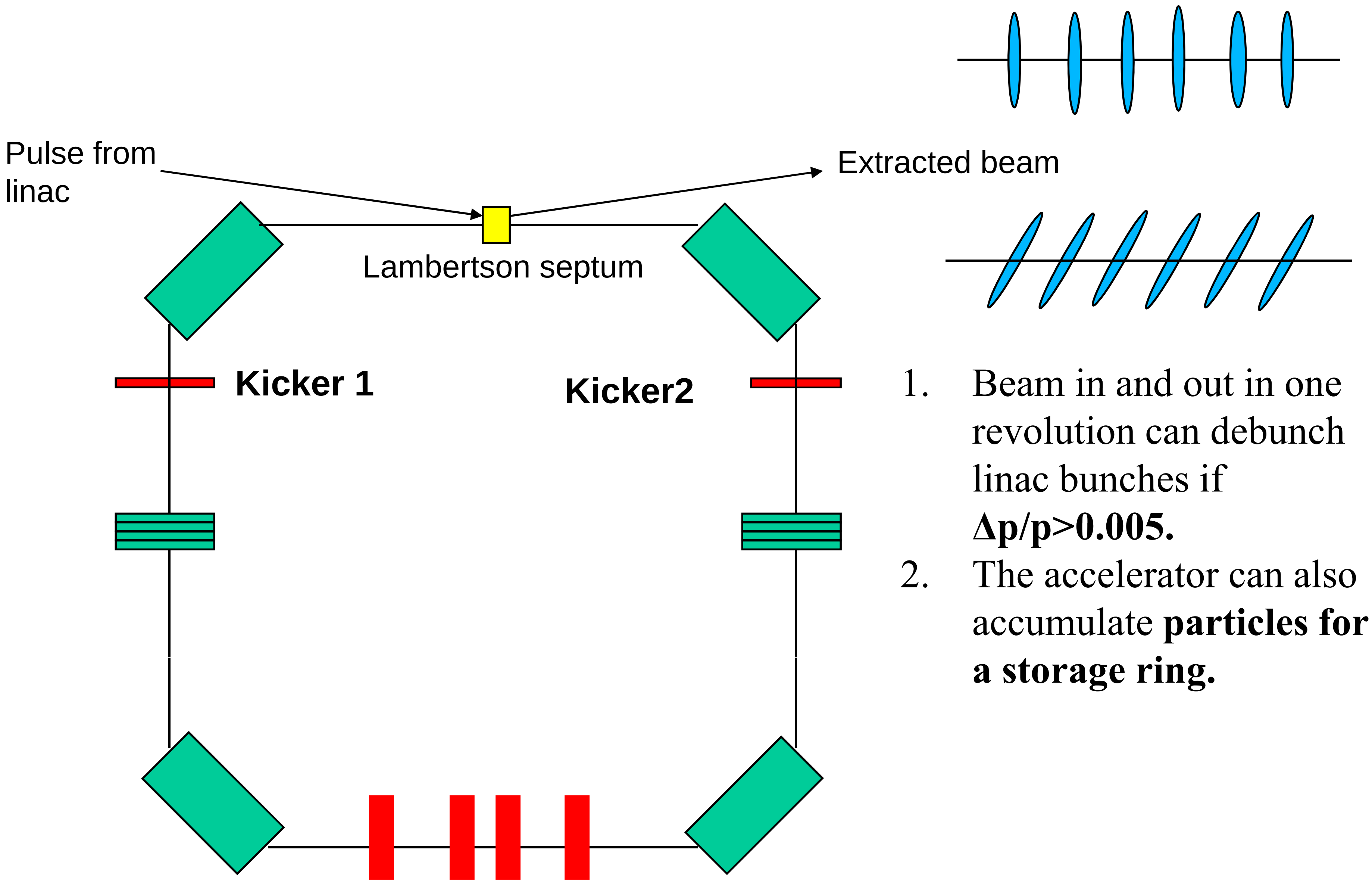
The separatrix orbits for η > 0 (above transition energy) with φ_s=2π/3, 5π/6, π, (top) and for η < 0 (below transition energy) with φ_s= 0, π/6, π/3 (bottom). The phase space area enclosed by the separatrix is called the bucket area. The stationary buckets that have largest phase space areas correspond to φ_s=0 (bottom) and π (top) respectively.

Summary:

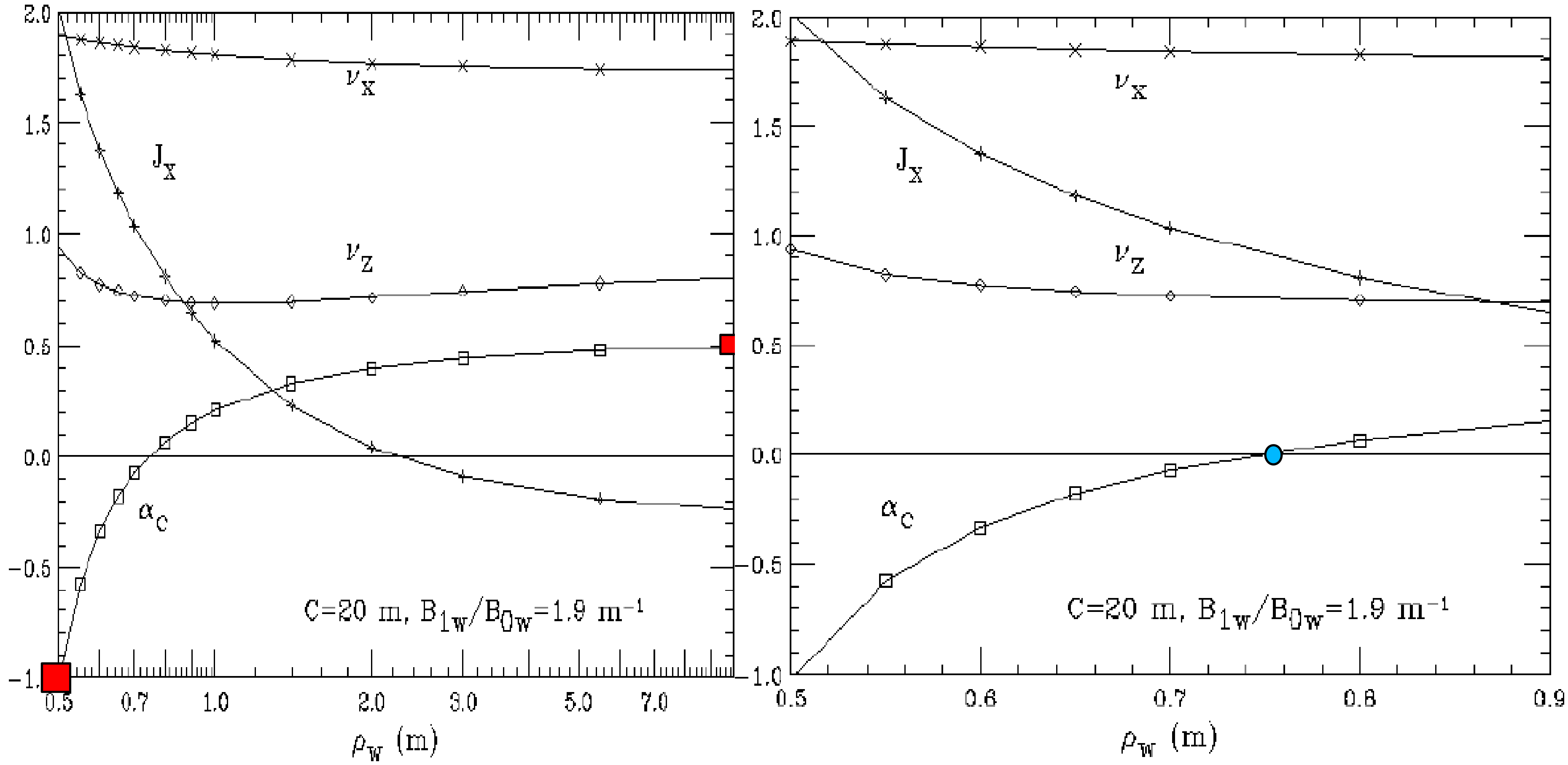
1. Particle motion in an accelerator can be described by 3D simple harmonic motion. The transverse degree of freedom is called **betatron motion** and the longitudinal degree of freedom is called the **synchrotron motion**.
2. The **betatron tunes** are number of betatron oscillations per revolution, and the synchrotron tune is the number of synchrotron oscillations per period. The betatron tunes increase with the size of the accelerator, while the synchrotron tune is about 10^{-4} to 10^{-2}.
3. The momentum compaction factor plays an important role in the accelerator. Typically, the momentum compaction factor for FODO cell lattice is $\boldsymbol{\alpha_c \sim 1/\nu_x^2}$. Thus the transition energy is $\gamma_T \sim \nu_x$. However, the momentum compaction for accelerators can be changed by changing the dispersion function in dipoles.

$$\Delta C = \oint \frac{D}{\rho} ds\,\delta, \qquad \alpha_{\mathrm{c}} \equiv \frac{d\Delta C}{C d\delta} = \frac{1}{C}\oint \frac{D}{\rho} ds \cong \frac{1}{C}\sum_i \langle D_i \rangle \theta_i$$

Examples in design of synchrotrons

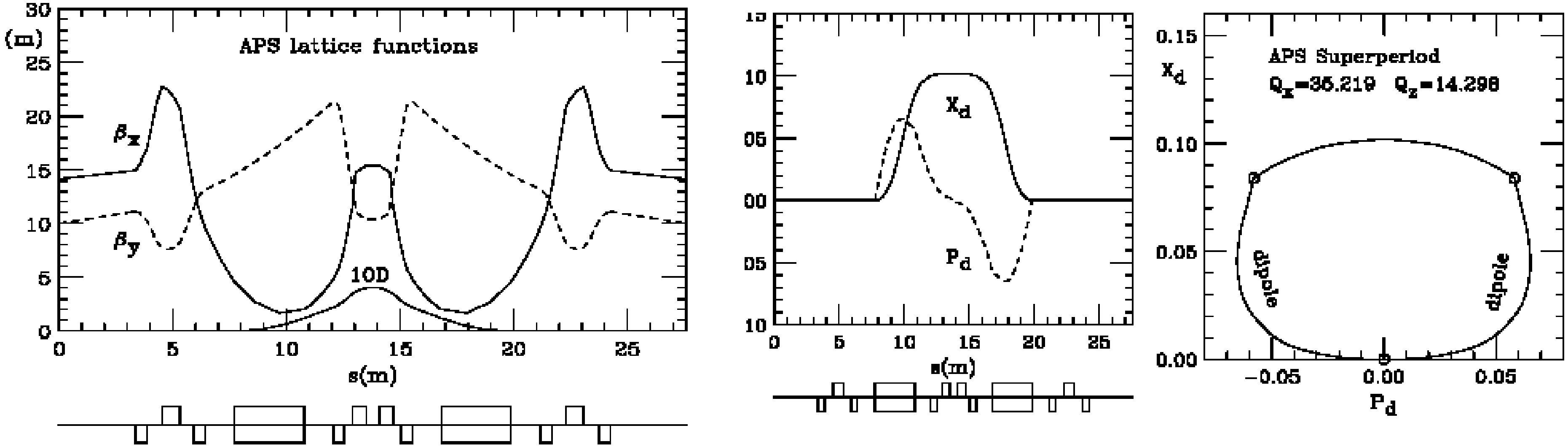


1. Beam in and out in one revolution can debunch linac bunches if **Δp/p>0.005.**
2. The accelerator can also accumulate **particles for a storage ring.**



Note that a large compaction factor is necessary for achieving de-bunching for the electron beams in a single path!

Example: APS lattice is made of 40 Double-bend Achromats (DBA) with a total length of 1104m. The momentum compaction factor for all DBA lattice is $\boldsymbol{\alpha_c = \rho\theta^2/(6R)}$. Because of its simplicity and flexibility, DBA lattice is commonly used as basic cells of synchrotron light source design.



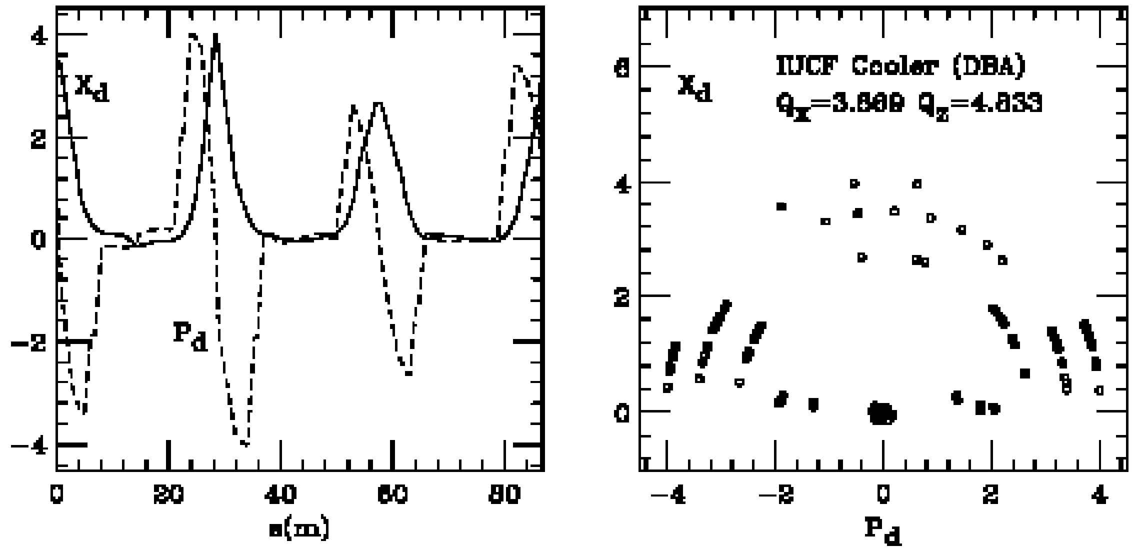
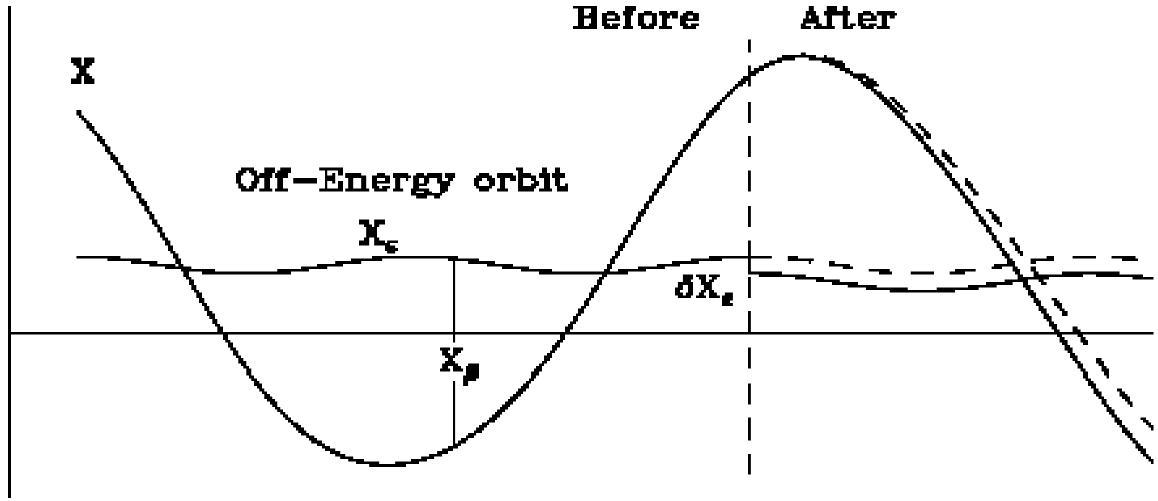


Figure 2.28: Left: Normalized dispersion phase-space coordinates X_d and P_d of the IUCF Cooler lattice are plotted. Right: The coordinates are shown in X_d vs P_d at the end of each lattice elements. The scales for both X_d and P_d are $m^{1/2}$.

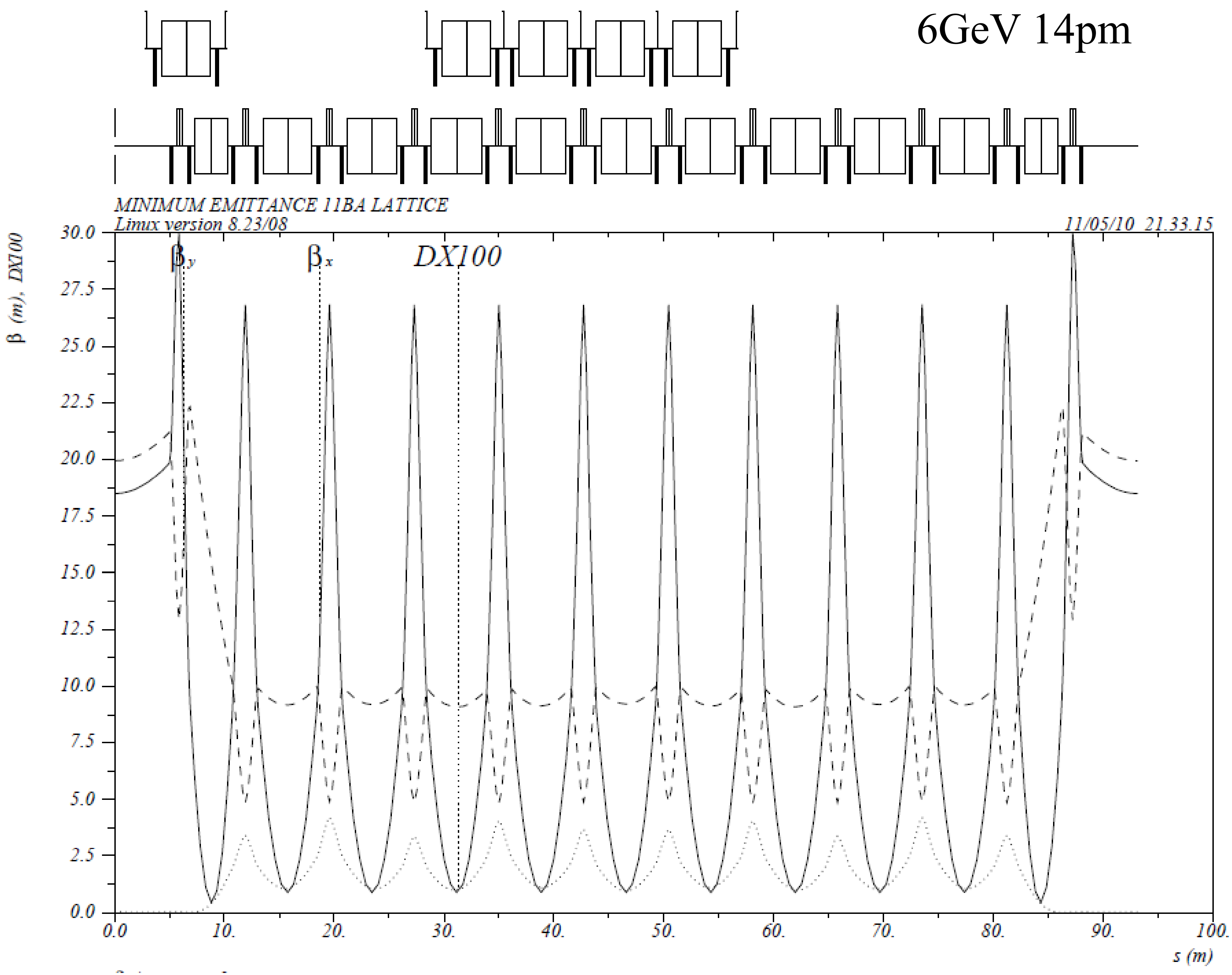
Dispersion function plays a very important role in the performance of high energy and synchrotron light source accelerators. For the synchrotron light source, the H-function plays a particular important role in determining the natural emittance of electron beams, i.e.

$$\varepsilon_x = F C_q \gamma^2 \theta^3, \quad C_q = \frac{55\hbar}{32\sqrt{3}mc} = 3.83\times 10^{-13}\,\mathrm{m}$$

The factor F is lattice dependent factor, F~1 for FODO cell, F~1/(4√15) for DBA lattice and F~1/(12√15) for the minimum emittance lattice.



	Colliders					Light Sources	
	BEPC	CESR	LER(e^+)	HER(e^-)	LEP	APS	ALS
E [GeV]	2.2	6	3.1	9	55	7	1.5
ν_x	5.8	9.38	32.28	25.28	76.2	35.22	14.28
ν_z	6.8	9.36	35.18	24.18	70.2	14.3	8.18
ρ [m]	10.35	60	30.6	165.0	3096.2	38.96	4.01
α [$\times 10^{-4}$]	400	152	14.9	24.4	3.866	2.374	14.3
C [m]	240.4	768.4	2199.3	2199.3	26658.9	1104	196.8
h	160	1281	3492	3492	31320	1296	328
f_{rf} [MHz]	199.5	499.8	476	476	352.2	352.96	499.65
ν_s	0.016	0.064	0.034	0.0522	0.085	0.006	0.0082
$\frac{\Delta E}{E_0}$ [$\times 10^{-4}$]	4.0	6.3	9.5	6.1	8.4	9.6	7.1
$\mathcal{A}$ [$\times 10^{-4}$eV-s]	3.5	7.2	3.1	5.7	78.	4.1	0.43
ϵ_x [nm]	450	240	64	48	51	8	4.8
ϵ_z [nm]	35	8	3.86	1.93	0.51	0.08	0.48

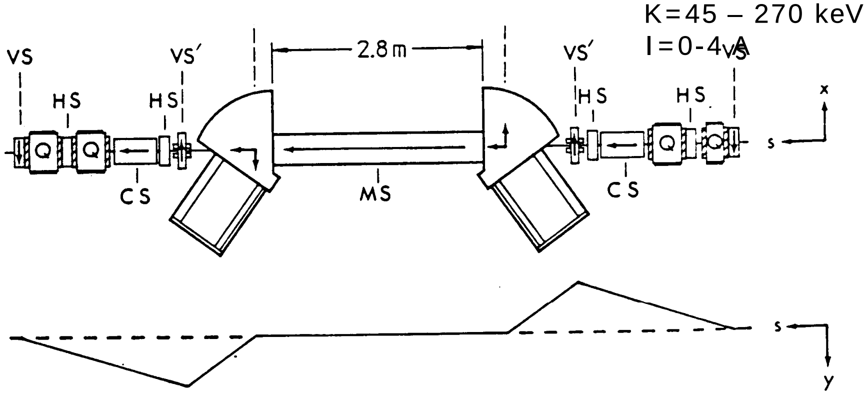

6GeV 14pm
MINIMUM EMITTANCE 11BA LATTICE
Linux version 8.23/08
11/05/10 21.33.15
β (m), DX100
βy
βx
DX100
s (m)
30.0
27.5
25.0
22.5
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
10.
20.
30.
40.
50.
60.
70.
80.
90.
100.

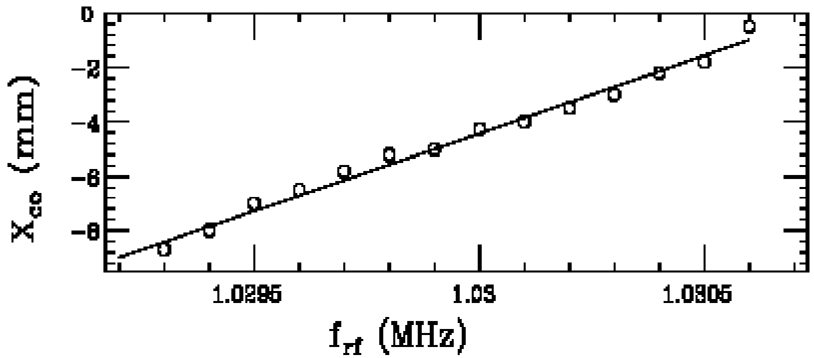
How to measure D(s)? $x_{co}(s)=D(s)\delta$

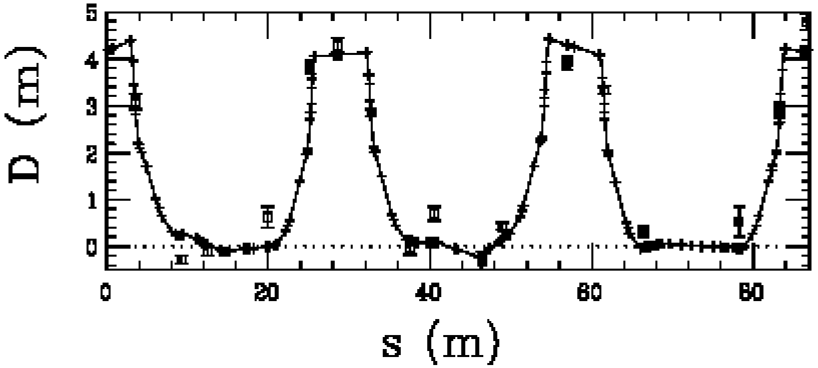
$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C} - \frac{\Delta v}{v} = (\alpha_c - \frac{1}{\gamma^2})\frac{\Delta p}{p_0} = \eta\delta,$$

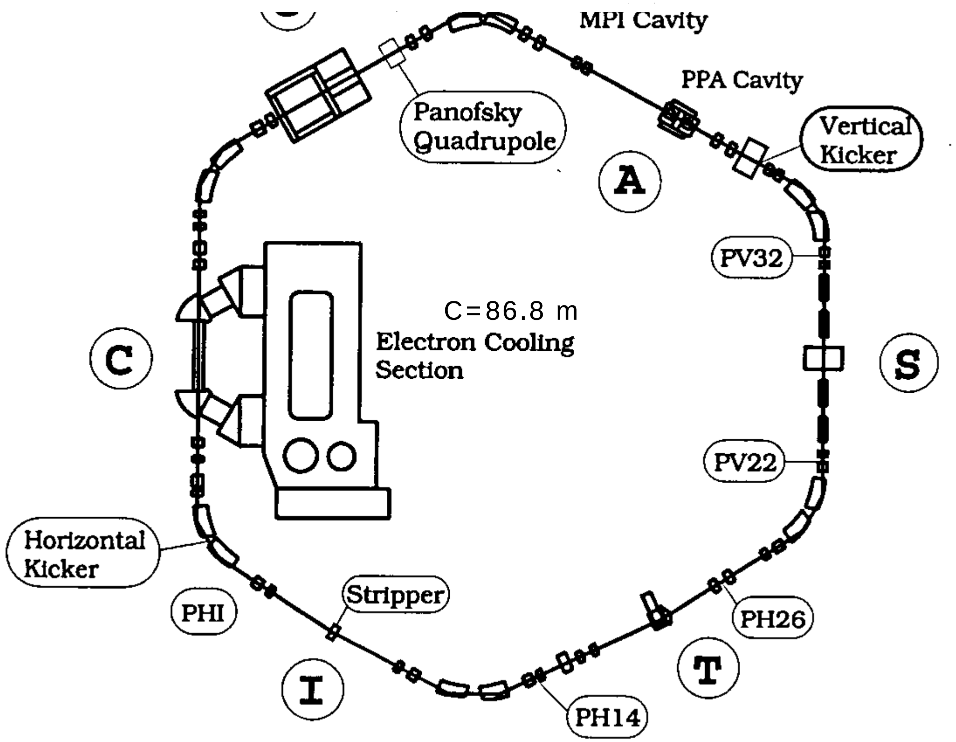
$$\Delta f / f_0 = -\eta\delta,$$

$$D = \frac{dx_{co}}{d(\Delta p/p_0)} = -\eta\, f_0 \frac{dx_{co}}{df_0},$$









Homework#3: $$\epsilon_{\rm rms} = \sqrt{\sigma_y^2 \sigma_{y'}^2 - \sigma_{yy'}^2} = \sigma_y \sigma_{y'} \sqrt{1 - r^2}. \tag{2.71}$$

The displacement vector from a reference orbit for a particle is $x = x_\beta + D\delta$, and $x = x_\beta + D\delta$, where x_β, x'_β are phase space coordinates of betatron motion and δ is the off-momentum parameter parameter. If the betatron motion and synchrotron motion are independent, show that (see Exeercise 2.2.14):

$$\sigma_x^2 = \beta_x \epsilon_x + D^2 \sigma_\delta^2, \quad \sigma_{x'}^2 = \gamma_x \epsilon_x + D'^2 \sigma_\delta^2, \quad \sigma_{x,x'} = -\alpha_x \epsilon_x + DD' \sigma_\delta^2,$$

where $\alpha_x(s)$, $\beta_x(s)$ and $\gamma_x(s)$ are the betatron amplitude functions, $D(s)$ and $D'(s)$ are dispersion functions, and ϵ_x and σ_δ are the rms emittance and the rms off-momentum width of the beam. Show also that the effective emittance defined as that of Eq. (2.71) is $\epsilon_{x,\rm eff} \equiv \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\epsilon_x \left[\epsilon_x + \mathcal{H}(s)\sigma_\delta^2\right]}$, where $\mathcal{H}(s) = \gamma_x D^2 + 2\alpha_x DD' + \beta_x D'^2$.

Exercise 2.2.14:

$$\langle y \rangle = \int y\rho(y, y')dydy', \quad \langle y' \rangle = \int y'\rho(y, y')dydy',$$

$$\sigma_y^2 = \int (y - \langle y \rangle)^2 \rho(y, y')dydy', \quad \sigma_{y'}^2 = \int (y' - \langle y' \rangle)^2 \rho(y, y')dydy',$$

$$\sigma_{yy'} = \int (y - \langle y \rangle)(y' - \langle y' \rangle)\rho(y, y')dydy' = r\sigma_y \sigma_{y'}, \quad \begin{pmatrix} \sigma_x^2 & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'}^2 \end{pmatrix} = \epsilon_{\rm rms} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

Solution: With dispersion function and off-momentum δ, the horizontal displacement from the center orbit is

$$x = x_\beta + D\delta, \quad \text{and} \quad x' = x'_\beta + D'\delta$$

If the betatron motion and synchrotron motion are independent, i.e. $\langle x_\beta \delta \rangle = 0$ and $\langle x'_\beta \delta \rangle = 0$, we find

$$\langle x^2 \rangle = \sigma^2_{\beta,xx} + D^2 \sigma^2_\delta$$
$$\langle x'^2 \rangle = \sigma^2_{\beta,x'x'} + D'^2 \sigma^2_\delta$$
$$\langle x x' \rangle = \sigma^2_{\beta,xx'} + D D' \sigma^2_\delta$$

Define effective emittance as $\quad \epsilon^2_{\text{eff}} = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2$

Use the result: $$\begin{pmatrix} \sigma^2_{x_\beta} & \sigma_{x_\beta x'_\beta} \\ \sigma_{x_\beta x'_\beta} & \sigma^2_{x'_\beta} \end{pmatrix} = \epsilon_{\text{rms}} \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

We find $$\epsilon_{x,\text{eff}} \equiv \sqrt{\sigma^2_x \sigma^2_{x'} - \sigma^2_{xx'}} = \sqrt{\epsilon_x \left[\epsilon_x + \mathcal{H}(s) \sigma^2_\delta\right]}$$

$$\mathcal{H}(s) = \gamma_x D^2 + 2\alpha_x D D' + \beta_x D'^2$$

Chromatic aberration

Inhomogeneous equation

$$x'' - \frac{\rho + x}{\rho^2} = \pm \frac{B_z}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2, \qquad z'' = -\frac{B_x}{B\rho}\frac{p_0}{p}(1+\frac{x}{\rho})^2.$$

$$p / p_0 = 1 + \delta$$

$$x'' + \left(\frac{1-\delta}{\rho^2(1+\delta)} - \frac{K(s)}{1+\delta}\right)x = \frac{\delta}{\rho(1+\delta)}, \quad K(s) = \frac{B_1}{B\rho}, \quad B_1 = \frac{\partial B_z}{\partial x}$$

$$x = x_\beta + D\delta \qquad D'' + \left(K_x(s) + \Delta K_x\right)D = \frac{1}{\rho} + O(\delta)$$

$$x_\beta'' + \left(K_x(s) + \Delta K_x\right)x_\beta = 0, \quad z_\beta'' + \left(K_z(s) + \Delta K_z\right)x_\beta = 0$$

$$K_x(s) = \frac{1}{\rho^2} - K(s), \quad \Delta K_x(s) = [-\frac{2}{\rho^2} + K(s)]\delta \approx -K_x(s)\delta,$$

$$K_z(s) = +K(s), \quad \Delta K_z(s) = [-K(s)]\delta = -K_z(s)\delta$$

Note that the betatron motion for off momentum particle is perturbed by a chromatic term. The betatron tunes must avoid half-integer resonances. But, the quadrupole error is proportional to the designed quadrupole field. They are called systematic chromatic aberration. It is an important topic in accelerator physics.

1. Tune shift, or tune spread, due to chromatic aberration:

$$\Delta\nu_x = \left[-\tfrac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \equiv C_x\delta, \quad C_x = d\nu_x / d\delta$$

$$\Delta\nu_z = \left[-\tfrac{1}{4\pi}\oint \beta_z(s)K_z(s)ds\right]\delta \equiv C_z\delta, \quad C_z = d\nu_z / d\delta$$

The chromaticity induced by quadrupole field error is called natural chromaticity. For a simple FODO cell, we find

$$\Delta\nu_x = \left[-\tfrac{1}{4\pi}\oint \beta_x(s)K_x(s)ds\right]\delta \approx -\delta\tfrac{1}{4\pi}\sum \beta_{xi}/f_i$$

$$C_{y,\mathrm{nat}}^{\mathrm{FODO}} = -\tfrac{1}{4\pi}N\left(\frac{\beta_{\max}}{f} - \frac{\beta_{\min}}{f}\right) = -\frac{\tan(\Phi/2)}{\Phi/2}\nu_y \approx -\nu_y$$

We define the specific chromaticity as

$$\xi_x = C_x/\nu_x, \qquad \xi_z = C_z/\nu_z$$

The specific chromaticity is about -1 for FODO cells, and can be as high as -4 for high luminosity colliders and high brightness electron storage rings.

$$\sin\frac{\Phi}{2} = \frac{L_1}{2f} \qquad \beta_{\max} = \frac{2L_1(1+\sin(\Phi/2))}{\sin\Phi}, \quad \beta_{\min} = \frac{2L_1(1-\sin(\Phi/2))}{\sin\Phi}$$

2. Chromaticity correction:

The chromaticity can cause tune spread to a beam with momentum spread $\Delta\nu = C\delta$. For a beam with C=-100, δ=0.005, $\Delta\nu$=0.5. The beam is not stable for most of the machine operation. Furthermore, there exists collective (head-tail) instabilities that requires positive chromaticity for stability! To correct chromaticity, we need to find magnetic field that provide stronger focusing for off-(higher)-momentum particles. We first try sextupole with $\Delta B_z + j\Delta B_x = B_0 b_2 (x + jz)^2, \quad A_s = \frac{1}{3}\mathrm{Re}\left\{B_0 b_2 (x + jz)^3\right\}$

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$

$$x = x_\beta + D\delta \qquad \Delta B_z = B_0 b_2 (x^2 - z^2) = B_0 b_2 (2x_\beta D\delta + D^2\delta^2 + x_\beta^2 - z_\beta^2)$$

$$z = z_\beta \qquad \Delta B_x = B_0 b_2 2xz = B_0 b_2 2z_\beta D\delta + B_0 b_2 2x_\beta z_\beta$$

Let $K_2 = -2B_0 b_2/B\rho = -B_2/B\rho$, we obtain:

$$x_\beta'' + (K_x(s) + K_2 D\delta)x_\beta = 0, \quad z_\beta'' + (K_z(s) - K_2 D\delta)z_\beta = 0$$

With sextupoles, the chromaticities becomes

$$C_x = -\tfrac{1}{4\pi}\oint \beta_x(s)[K_x(s) - K_2(s)D(s)]ds$$

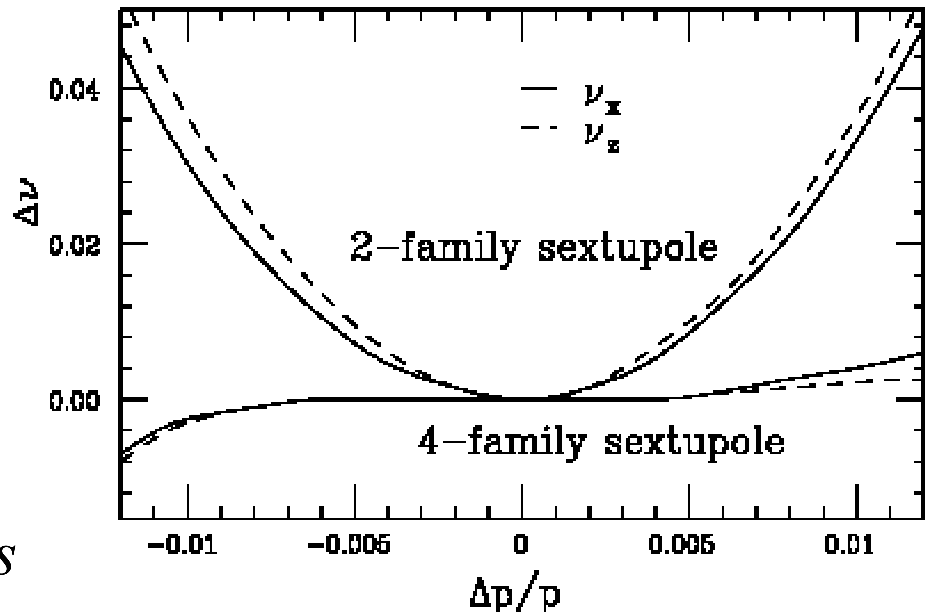
$$C_z = -\tfrac{1}{4\pi}\oint \beta_z(s)[K_z(s) + K_2(s)D(s)]ds$$

For FODO cells, the integrated sextupole strength is

$$S_{\mathrm{F}} \equiv K_2\ell_{\mathrm{SF}} = \frac{\sin(\Phi/2)}{2f^2\theta(1+\frac{1}{2}\sin(\Phi/2))}, \quad S_{\mathrm{D}} \equiv K_{2\mathrm{D}}\ell_{\mathrm{SD}} = -\frac{\sin(\Phi/2)}{2f^2\theta(1-\frac{1}{2}\sin(\Phi/2))}$$

For high energy colliders and high brightness synchrotron light sources, the sextupole strength can be much higher. Even more important is the effect of the systematic half-integer stopbands.

$$J_{y,p} = \frac{1}{2\pi}\oint \left[\beta_y(s)\Delta K_y(s)\right] e^{-jp\phi_y(s)} ds$$



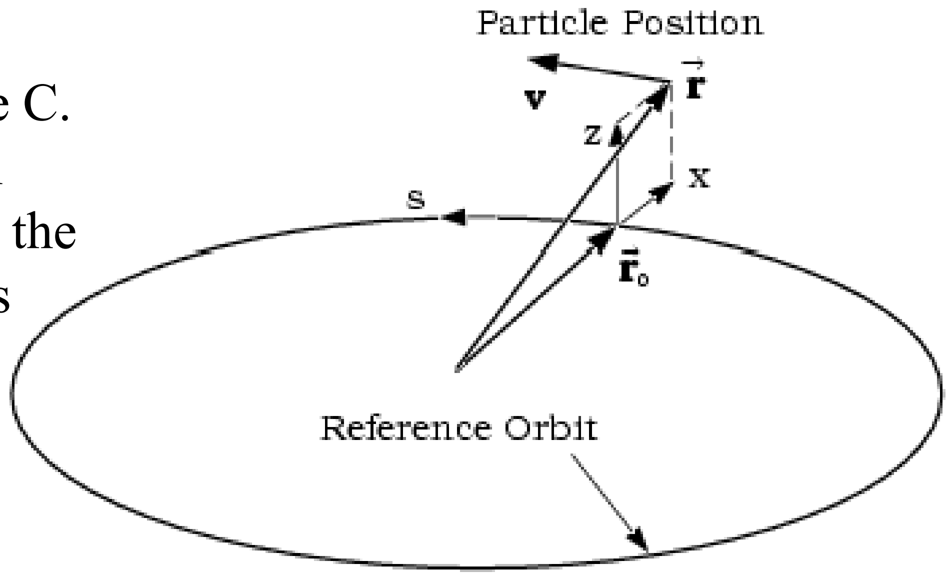
Synchrotorn Motion

A review:

$$H = e\Phi + c\left[m^2c^2 + \frac{(p_s - eA_s)^2}{(1+x/\rho)^2} + (p_x - eA_x)^2 + (p_z - eA_z)^2 \right]^{1/2}$$

$$\dot{s} = \frac{\partial H}{\partial p_s}, \dot{p}_s = -\frac{\partial H}{\partial s}; \quad \dot{x} = \frac{\partial H}{\partial p_x}, \dot{p}_x = -\frac{\partial H}{\partial x}; \quad \dot{z} = \frac{\partial H}{\partial p_z}, \dot{p}_z = -\frac{\partial H}{\partial z}.$$

The phase space coordinates are (*x,s,z*) with independent coordinate *t*. In one revolution, the time advances T_0, called the orbital period. In one orbital period, the particle orbit is equal to the circumference C. All accelerator components repeat in each orbital period. It would be nice to use *s* as the independent coordinate. How to make this coordinate transfer?



$$x' = \frac{dx}{ds} = \frac{\dot{x}}{\dot{s}} = \left(\frac{\partial H}{\partial p_x}\right)\left(\frac{\partial H}{\partial p_s}\right)^{-1} = \frac{\partial(-p_s)}{\partial p_x},$$

$$dH = (\partial H/\partial p_x)dp_x + (\partial H/\partial p_s)dp_s = 0$$

$$t' = \frac{\partial p_s}{\partial H}, H' = -\frac{\partial p_s}{\partial t}; \quad x' = -\frac{\partial p_s}{\partial p_x}, p_x' = \frac{\partial p_s}{\partial x}; \quad z' = -\frac{\partial p_s}{\partial p_z}, p_z' = \frac{\partial p_s}{\partial z}.$$

These equations indicate that $-p_s$ becomes the new Hamiltonian with the ($x,p_x,z,p_z,$**t,-H**) and *s* as the independent coordinate.

$$\tilde{H} = -\left(1+\frac{x}{\rho}\right)\left[\frac{(H-e\phi)^2}{c^2} - m^2c^2 - (p_x - eA_x)^2 - (p_z - eA_z)^2\right]^{1/2} - eA_s,$$

$$\widetilde{H} \approx -p(1+\frac{x}{\rho}) + \frac{1+x/\rho}{2p}[(p_x - eA_x)^2 + (p_z - eA_z)^2] - eA_s$$

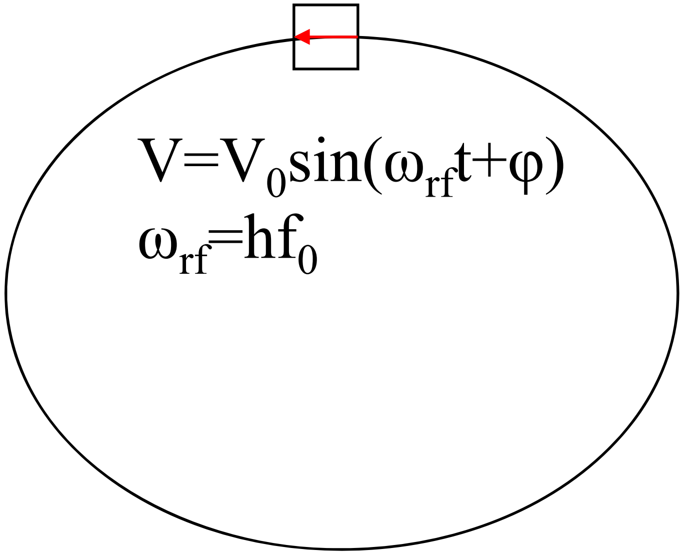
$$x' = \frac{\partial \widetilde{H}}{\partial p_x}, \ p_x' = -\frac{\partial \widetilde{H}}{\partial x}, \quad z' = \frac{\partial \widetilde{H}}{\partial p_z}, \ p_z' = -\frac{\partial \widetilde{H}}{\partial z}, \quad t' = \frac{\partial \widetilde{H}}{\partial H}, \quad -H' = -\frac{\partial \widetilde{H}}{\partial t}.$$

$$x'' + K_x(s)x = \frac{\Delta B_z}{B\rho}, \quad z'' + K_z(s)z = -\frac{\Delta B_x}{B\rho}$$ Hill's equation

$$\Delta E_{n+1} = \Delta E_n + eV(\sin\phi_n - \sin\phi_s), \quad \phi_{n+1} = \phi_n + \frac{2\pi\eta}{\beta^2 E}\Delta E_{n+1}$$ Synchrotron motion

$$\eta = \alpha_{\mathrm{c}} - \frac{1}{\gamma^2} = \frac{1}{\gamma_{\mathrm{T}}^2} - \frac{1}{\gamma^2}.$$

$$T = \frac{C}{\upsilon}, \quad \frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta \upsilon}{\upsilon} = \left(\alpha_c - \frac{1}{\gamma^2} \right) \frac{\Delta p}{p_0} = \eta \delta$$



$$R = R_0 (1 + \alpha_0 \delta + \alpha_1 \delta^2 + \alpha_2 \delta^3 + \ldots)$$

$$\alpha_{\mathrm{c}} = \frac{dR}{R_0 d\delta} = \alpha_0 + 2\alpha_1 \delta + 3\alpha_2 \delta^2 + \ldots \equiv \frac{1}{\gamma_{\mathrm{T}}^2}$$

$$\frac{\Delta \omega}{\omega_0} = -\eta(\delta)\delta = -(\eta_0 + \eta_1 \delta + \eta_2 \delta^2 + \ldots)\delta$$

$$\eta_0 = (\alpha_0 - \frac{1}{\gamma_0^2}), \quad \eta_1 = \frac{3\beta_0^2}{2\gamma_0^2} + \alpha_1 - \alpha_0 \eta_0, \ \ldots\ldots.$$

For the moment, we neglect all nonlinear effects in the phase slip factor. The equation of motion is given by

$$\frac{d}{dt}(\phi - \phi_{\mathrm{s}}) = -h\Delta\omega = h\omega_0 \frac{\Delta T}{T_0} = h\eta\omega_0 \frac{\Delta p}{p_0} = \frac{\eta h \omega_0^2}{\beta^2 E_0} \frac{\Delta E}{\omega_0}.$$

$$\frac{d}{dt}\left(\frac{\Delta E}{\omega_0}\right) = \frac{1}{2\pi} e V_0 (\sin\phi - \sin\phi_{\mathrm{s}}),$$

Fundamentals of rf systems:

Synchrotrons require rf cavities for particle acceleration. The rf cavities are devices that can hold strong electromagnetic field. RF cavity has characteristic of (1) **frequency ω_{rf}, harmonic number**, (2) **V_{rf}, shunt impedance, Q-factor, and transit time factor.**

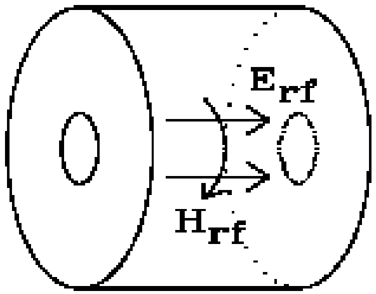
The cavity design is given in another class. Let the cavity gap be g, electric field amplitude be E, and speed of the particle be v, the energy gain in a cavity with sinusoidal varying electric field is reduced by a transit time factor.

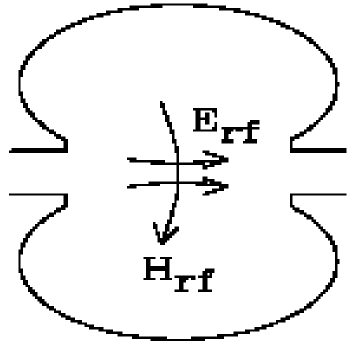
We normally include the transit time factor in the voltage amplitude of the accelerator rf cavity.

$$\Delta E = e\int_{-g/2}^{g/2} \mathrm{E}\cos\frac{\omega s}{\upsilon}ds = e\mathrm{E}gT_{\mathrm{tr}}, \quad T_{\mathrm{tr}} = \frac{\sin(\pi \mathrm{g}/\beta\lambda)}{(\pi \mathrm{g}/\beta\lambda)}$$

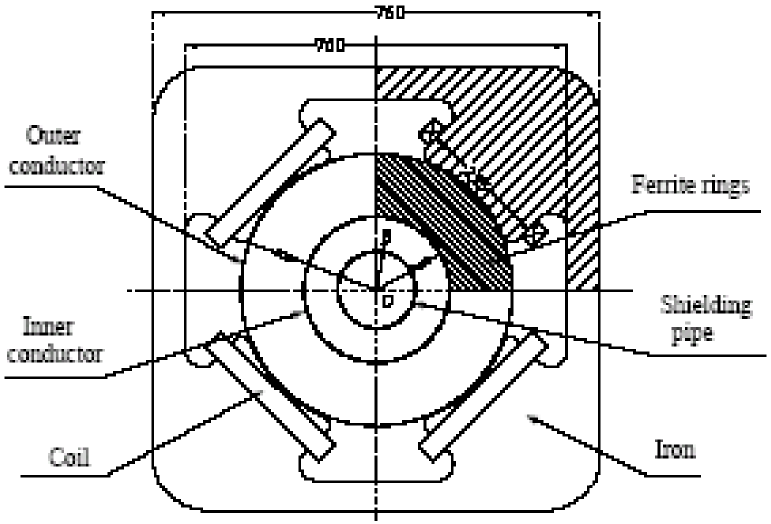
An rf cavity is a device to store electromagnetic energy at a particular frequency with minimum energy loss. The cavity can be designed with many different shapes and geometry.

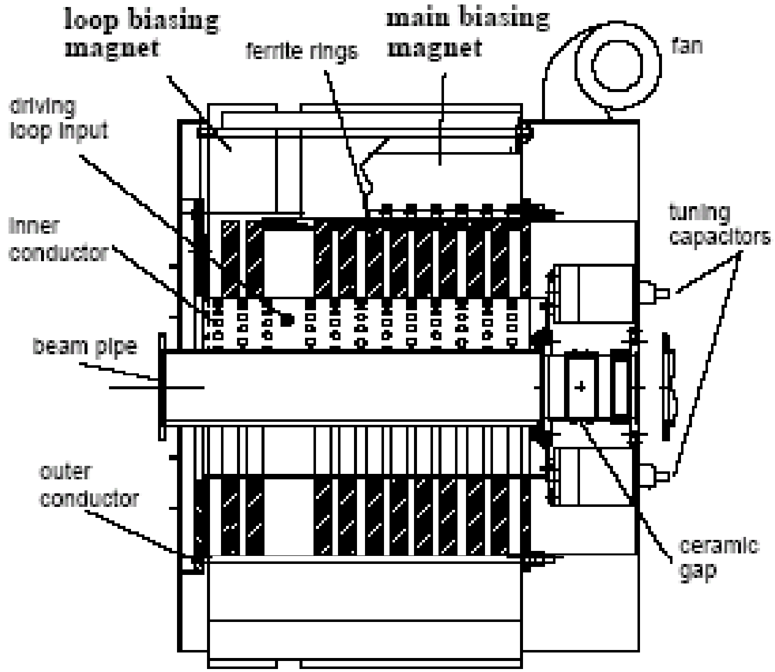
Pill-box cavity





Ferrite loaded cavity





	proton	electron
KE (GeV)	0.3	2.999
E (GeV)	1.238	3
p (GeV/c)	0.808	3
Brho (T-m)	2.695	10.007
BL (T-m)	16.936	62.875
B (T)	1.4	1.3
L(m)	12.097	48.366
N_dip	4	48
L_dip (m)	3.024	1.0076
P	2	24
f_{rf} (MHz)	**1.3-7.0**	**499.654**
C (m)	28.5	518.4
$L_{1/2}$ (m)	3.5	DBA
nu_x	1.68	26.24
nu_z	0.72	14.3

$$B\rho = 3.33564 \frac{A}{Z} p\,[\mathrm{GeV/c/u}]$$

$$\frac{\int B d\ell}{B\rho} = 2\pi,$$

$$\sum B_i \ell_i = 2\pi B\rho$$

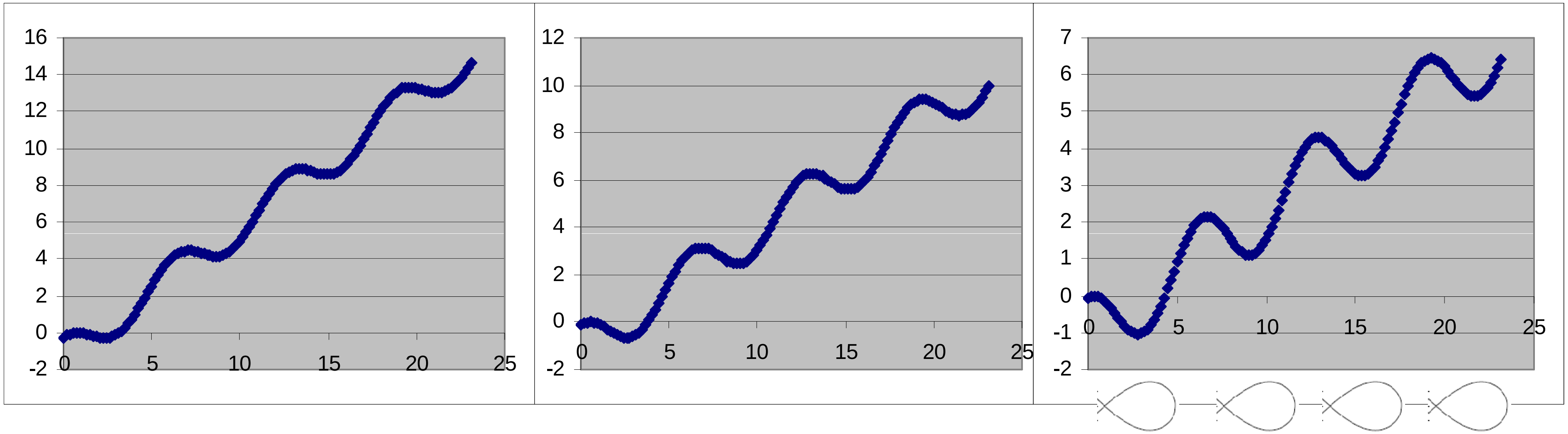
$$\omega_{\mathrm{rf}} = h\frac{\beta c}{R_0} = \frac{he\rho B}{R_0 \gamma m} = \frac{hc}{R_0}\left[\frac{B^2(t)}{B^2(t) + (mc^2/ec\rho)^2}\right]^{1/2}$$

$$\frac{mc^2}{ec\rho} = \begin{cases} 3.1273/\rho\,[\mathrm{m}]\ \mathrm{Tesla} & \text{for protons,} \\ 0.001703/\rho\,[\mathrm{m}]\ \mathrm{Tesla} & \text{for electrons,} \end{cases}$$

h=864

The synchrotron equation of motion can be derived from the Hamiltonian for phase space coordinates (ϕ,$\Delta E/\omega_0$):

$$H = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right)^2 + \frac{eV}{2\pi}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$



ϕ_s=45° ϕ_s=30° ϕ_s=20°

Note that the second term in the Hamiltonian can be visualized as the potential. Stable particle motion is bounded by the potential well. The area of stable motion is called bucket.

Equivalently, the synchrotron Hamiltonian for the phase space coordinates (ϕ,δ) is

$$H = \frac{1}{2} h\omega_0\eta_0\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E}[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s]$$

$$\dot{\phi} = h\eta\omega_0\delta \qquad\qquad \dot{\delta} = \frac{\omega_0 eV_0}{2\pi\beta^2 E}(\sin\phi - \sin\phi_s)$$

The fixed point of the Hamiltonian is located at phase space: $(\phi_s,0)$ and $(\pi-\phi_s,0)$. Small amplitude motion around the stable fixed point $(\phi_s,0)$ is nearly simple harmonic with synchrotron tune Q_s given by

$$Q_s = \sqrt{\frac{h\,|\eta\cos\phi_s\,|\,eV_0}{2\pi\beta^2 E}} = \nu_s\sqrt{|\cos\phi_s\,|} \qquad\qquad \nu_s = \sqrt{\frac{h\,|\eta|\,eV_0}{2\pi\beta^2 E}}$$

$$Q_s = \sqrt{heV|\eta_0\cos\phi_s|/2\pi\beta^2 E}$$

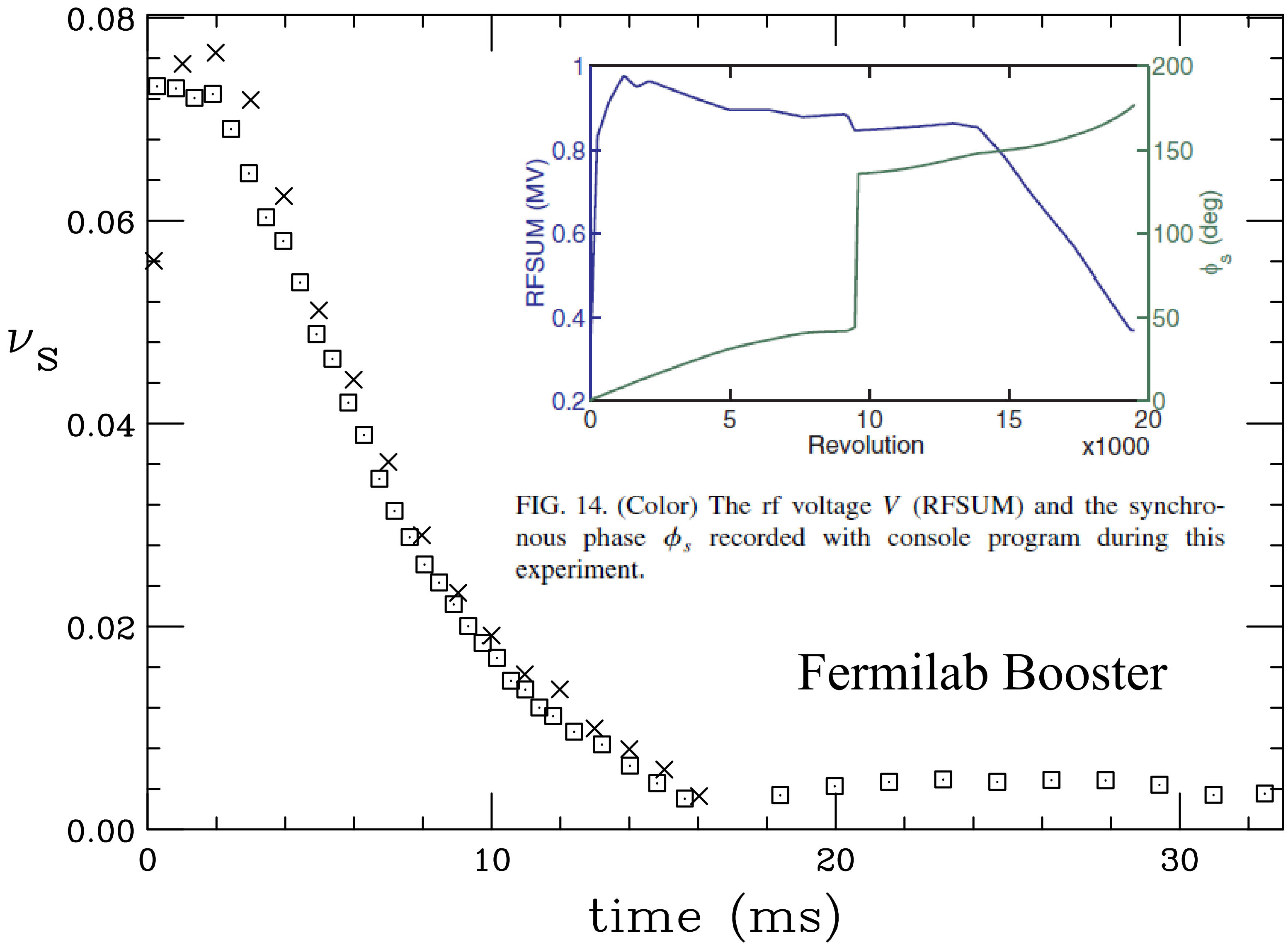
	AGS	RHIC	FNAL-MI	FNAL-BST	SSC	Cooler
$\nu_s(\times 10^{-3})$	32.3	0.589	13.7	18.1	0.839	0.395

Requirement of rf voltage in rapid accelerating accelerators

$$B\rho = \frac{p}{e}, \quad \dot{p} = \frac{1}{\beta c}\dot{E}, \quad f = \frac{\beta c}{2\pi R}, \quad \Longrightarrow \quad V \sin\phi_s = 2\pi R\rho\dot{B}.$$

For electron storage ring: V sin(φ_s)= energy loss per revolution

$$\omega_{\rm rf} = h\frac{\beta c}{R_0} = \frac{he\rho B}{R_0\gamma m} = \frac{hc}{R_0}\left[\frac{B^2(t)}{B^2(t) + (mc^2/ec\rho)^2}\right]^{1/2}$$



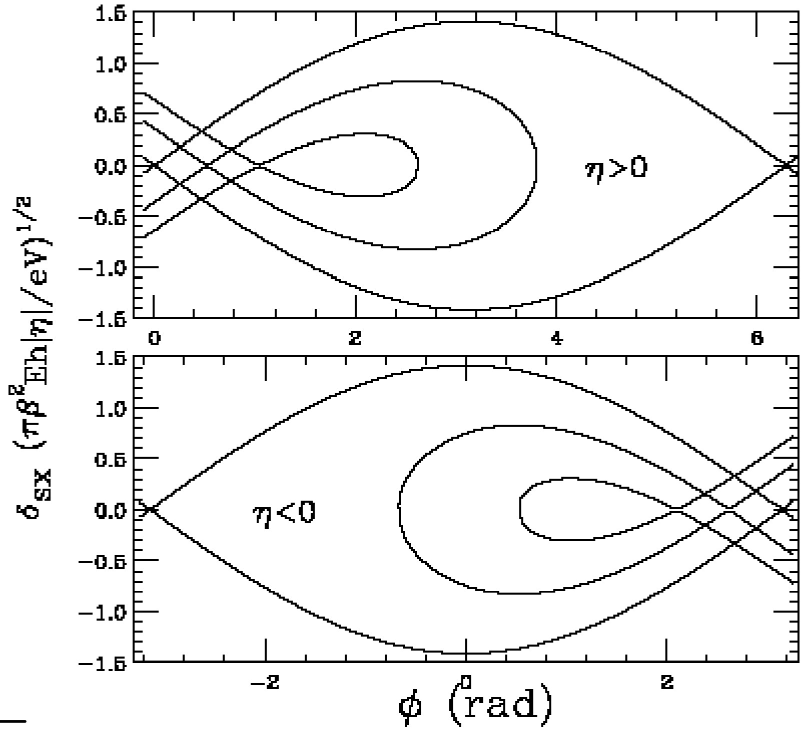
The synchrotron tune in a booster cycle. The squares are measured from turn-by-turn data with ICA method. The crosses are measured from phase signal with synchrotron phase detector (SPD). Note that the SPD method has difficulty in measuring the synchrotron tune above the transition energy at around the 14.5 ms.

The Hamiltonian torus that passes through the unstable fixed point is called the separatrix.

$$H(\phi,\delta)=H_{\mathrm{sx}}=H(\pi-\phi_{\mathrm{s}},0)=\frac{\omega_0 eV_0}{2\pi\beta^2 E}[-2\cos\phi_{\mathrm{s}}+(\pi-2\phi_{\mathrm{s}})\sin\phi_{\mathrm{s}}]$$

$$\delta_{\mathrm{sx}}^2+\frac{eV_0}{\pi\beta^2 Eh\eta}[\cos\phi+\cos\phi_{\mathrm{s}}-(\pi-\phi-\phi_{\mathrm{s}})\sin\phi_{\mathrm{s}}]=0$$

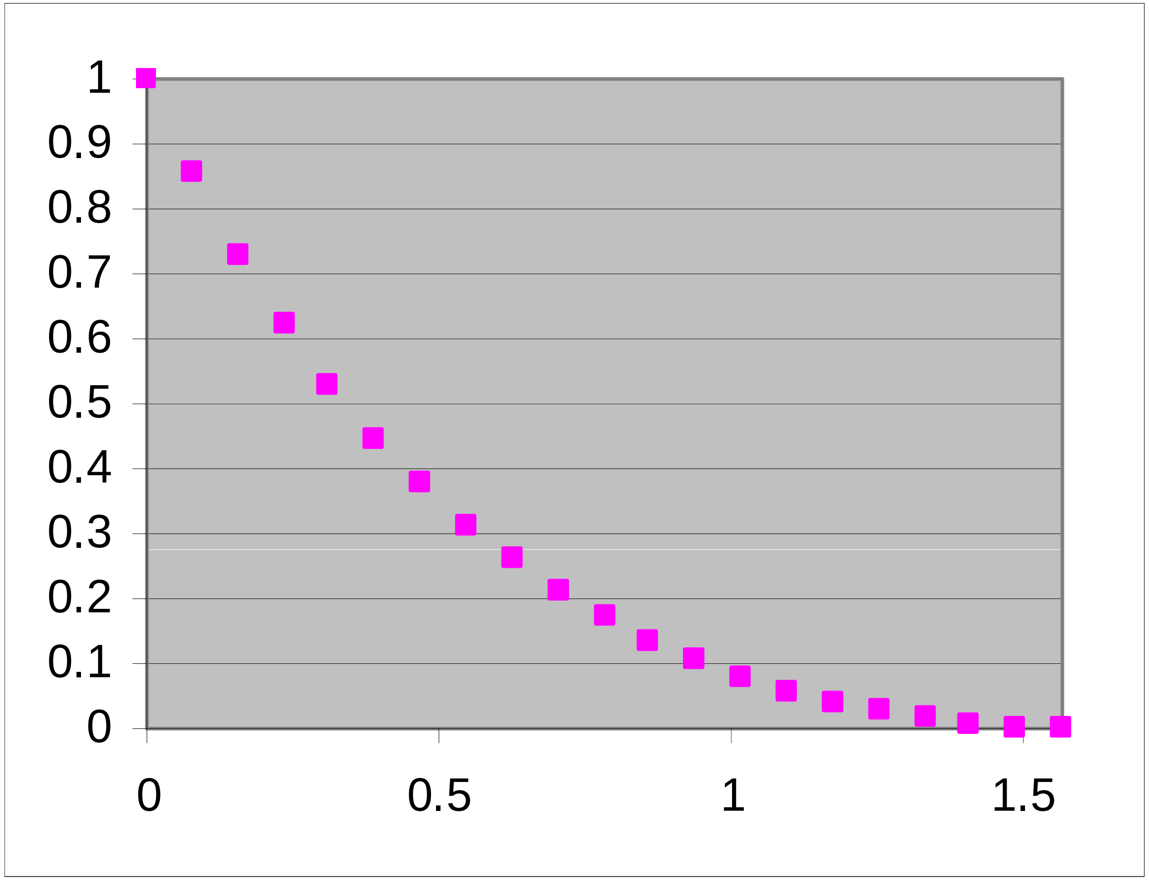
The phase space area enclosed by the separatrix is called the bucket, where particle motion around the stable fixed point is elliptical. The motion around the unstable fixed point is hyperbolical. The bucket area is defined as



$$\widetilde{A}_{\mathrm{B}}=\oint\delta_{\mathrm{sx}}(\phi)d\phi=16\sqrt{\frac{eV_0}{2\pi\beta^2 Eh|\eta|}}\alpha_{\mathrm{b}}(\phi_{\mathrm{s}})=\frac{16\nu_{\mathrm{s}}}{h|\eta|\sqrt{|\cos\phi_{\mathrm{s}}|}}\alpha_{\mathrm{b}}(\phi_{\mathrm{s}})$$

The moving bucket factor is given by

$$\alpha_{\rm b}(\phi_{\rm s}) = \frac{1}{4\sqrt{2}} \int_{\phi_{\rm u}}^{\pi-\phi_{\rm s}} \left[-\frac{|\eta|}{\eta} \left[\cos\phi + \cos\phi_{\rm s} - (\pi - \phi - \phi_{\rm s}) \sin\phi_{\rm s} \right] \right]^{1/2} d\phi.$$



Φ_s (rad)

The bucket area in phase space ($\phi, \Delta E/\omega_0$) is given by

$$A_{\mathrm{B}} = \frac{\beta^2 E}{\omega_0} \widetilde{A}_{\mathrm{B}} = h \Delta t \Delta E$$

The phase space area measures the time-width, and energy-spread of the bunch distribution. Thus the dimension of the phase space area is eV-sec. For example, a beam bunch with 100 ns bunch length and 1 MeV energy spread have a bunch area of 0.1 eV-sec. A beam with 1 MeV energy spread with 1 GeV energy has a fractional energy spread of 10^{-3}.

	$(\phi, \frac{\Delta E}{\omega_0})$	(ϕ, δ)	$(\phi, \frac{h\|\eta\|}{\nu_s}\delta)$
Bucket Area	$16\left(\frac{\beta^2 EeV}{2\pi\omega_0^2 h\|\eta\|}\right)^{1/2} \alpha_b(\phi_s)$	$16\left(\frac{eV}{2\pi\beta^2 Eh\|\eta\|}\right)^{1/2} \alpha_b(\phi_s)$	$16\ \alpha_b(\phi_s)$
Bucket Height	$2\left(\frac{\beta^2 EeV}{2\pi\omega_0^2 h\|\eta\|}\right)^{1/2} Y(\phi_s)$	$2\left(\frac{eV}{2\pi\beta^2 Eh\|\eta\|}\right)^{1/2} Y(\phi_s)$	$2\ Y(\phi_s)$

Using time t as an independent variable, the equations of motion and the Hamiltonian are listed as follows.

- Using $(\phi, \Delta E/\omega_0)$ as phase-space coordinates:

$$\frac{d\phi}{dt} = \frac{h\omega_0^2\eta}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right), \quad \frac{d\,(\Delta E/\omega_0)}{dt} = \frac{1}{2\pi}eV(\sin\phi - \sin\phi_{\rm s}), \tag{3.33}$$

$$H = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2 E}\left(\frac{\Delta E}{\omega_0}\right)^2 + \frac{eV}{2\pi}[\cos\phi - \cos\phi_{\rm s} + (\phi - \phi_{\rm s})\sin\phi_{\rm s}] \tag{3.34}$$

- Using (ϕ, δ) as phase-space coordinates:

$$\frac{d\phi}{dt} = h\omega_0\eta\delta, \quad \frac{d\delta}{dt} = \frac{\omega_0 eV}{2\pi\beta^2 E}(\sin\phi - \sin\phi_{\rm s}), \tag{3.35}$$

$$H = \frac{1}{2}h\omega_0\eta\delta^2 + \frac{\omega_0 eV}{2\pi\beta^2 E}[\cos\phi - \cos\phi_{\rm s} + (\phi - \phi_{\rm s})\sin\phi_{\rm s}]. \tag{3.36}$$

- Using $(\phi, \mathcal{P} = -(h|\eta|/\nu_{\rm s})\delta)$ as the normalized phase-space coordinates:

$$\frac{d\phi}{dt} = \omega_0\nu_{\rm s}\mathcal{P}, \quad \frac{d\mathcal{P}}{dt} = \frac{\eta}{|\eta|}\omega_0\nu_{\rm s}(\sin\phi - \sin\phi_{\rm s}), \tag{3.37}$$

$$H = \frac{1}{2}\omega_0\nu_{\rm s}\mathcal{P}^2 + \frac{\eta}{|\eta|}\omega_0\nu_{\rm s}[\cos\phi - \cos\phi_{\rm s} + (\phi - \phi_{\rm s})\sin\phi_{\rm s}]. \tag{3.38}$$

- Using $(\tau = (\phi - \phi_{\rm s})/h\omega_0, \dot\tau)$ as phase-space coordinates:

$$\frac{d\tau}{dt} = \dot\tau, \quad \frac{d\dot\tau}{dt} = -\frac{\eta\omega_0 eV}{2\pi\beta^2 E}[\sin(\phi_{\rm s} - h\omega_0\tau) - \sin\phi_{\rm s}], \tag{3.39}$$

$$H = \frac{1}{2}\dot\tau^2 + \frac{\eta eV}{2\pi h\beta^2 E}[\cos(\phi_{\rm s} - h\omega_0\tau) - \cos\phi_{\rm s} - h\omega_0\tau\sin\phi_{\rm s}] \tag{3.40}$$

The corresponding normalized phase space is $(\tau, \dot\tau/\omega_{\rm s})$.

B. Using longitudinal distance s as independent variable

- Using $(R\phi/h, -\Delta p/p_0)$ as phase-space coordinates, the Hamiltonian is

$$H = -\frac{1}{2}\eta\left(\frac{\Delta p}{p_0}\right)^2 - \frac{\nu_{\rm s}^2}{h^2|\eta|}\left[\cos\phi - \cos\phi_{\rm s} + (\phi - \phi_{\rm s})\sin\phi_{\rm s}\right], \tag{3.41}$$

II.3 Small-Amplitude Oscillations and Bunch Area

The linearized synchrotron Hamiltonian around the SFP is

$$H = \frac{1}{2}h\omega_0\eta\delta^2 - \frac{\omega_0 eV\cos\phi_\mathrm{s}}{4\pi\beta^2 E}\varphi^2, \tag{3.53}$$

where $\varphi = \phi - \phi_\mathrm{s}$. The synchrotron frequency is given by Eq. (3.27), and

$$\varphi = \hat{\phi}\cos(\omega_\mathrm{s}t + \chi), \quad \delta = -\frac{Q_\mathrm{s}}{h|\eta|}\hat{\phi}\sin(\omega_\mathrm{s}t + \chi), \tag{3.54}$$

where $\omega_\mathrm{s} = Q_\mathrm{s}\omega_0$ is the angular synchrotron tune. The phase-space ellipse of a particle becomes

$$\left(\frac{\delta^2}{\hat{\delta}}\right)^2 + \left(\frac{\varphi}{\hat{\phi}}\right)^2 = 1, \quad \frac{\hat{\delta}}{\hat{\phi}} = \left(\frac{eV|\cos\phi_\mathrm{s}|}{2\pi\beta^2 Eh|\eta|}\right)^{1/2} = \frac{Q_\mathrm{s}}{h|\eta|}, \tag{3.55}$$

where $\hat{\delta}$ and $\hat{\phi}$ are maximum amplitudes of the phase-space ellipse. The phase-space area of the ellipse is $\pi\hat{\delta}\hat{\phi}$.

II.3 Small-Amplitude Oscillations and Bunch Area

The linearized synchrotron Hamiltonian around the SFP is

$$H = \frac{1}{2} h\omega_0 \eta \delta^2 - \frac{\omega_0 eV \cos\phi_s}{4\pi\beta^2 E}\varphi^2, \tag{3.53}$$

where $\varphi = \phi - \phi_s$. The synchrotron frequency is given by Eq. (3.27), and

$$\varphi = \hat{\phi}\cos(\omega_s t + \chi), \quad \delta = -\frac{Q_s}{h|\eta|}\hat{\phi}\sin(\omega_s t + \chi), \tag{3.54}$$

where $\omega_s = Q_s\omega_0$ is the angular synchrotron tune. The phase-space ellipse of a particle becomes

$$\left(\frac{\delta^2}{\hat{\delta}}\right)^2 + \left(\frac{\varphi}{\hat{\phi}}\right)^2 = 1, \quad \frac{\hat{\delta}}{\hat{\phi}} = \left(\frac{eV|\cos\phi_s|}{2\pi\beta^2 E h|\eta|}\right)^{1/2} = \frac{Q_s}{h|\eta|}, \tag{3.55}$$

where $\hat{\delta}$ and $\hat{\phi}$ are maximum amplitudes of the phase-space ellipse. The phase-space area of the ellipse is $\pi\hat{\delta}\hat{\phi}$.

For a given phase-space area A, the maximum fractional momentum **deviation** and phase-width are related to the area by

$$\hat{\delta} = \mathcal{A}^{1/2}\left(\frac{\omega_0}{\pi\beta^2 E}\right)^{1/2}\left(\frac{heV|\cos\phi_s|}{2\pi\beta^2 E|\eta|}\right)^{1/4},$$

$$\hat{\theta} = \frac{1}{h}\hat{\phi} = \mathcal{A}^{1/2}\left(\frac{\omega_0}{\pi\beta^2 E}\right)^{1/2}\left(\frac{2\pi\beta^2 E|\eta|}{heV|\cos\phi_s|}\right)^{1/4},$$

$$\frac{\hat{\delta}}{\hat{\theta}} = \left(\frac{heV|\cos\phi_s|}{2\pi\beta^2 E|\eta|}\right)^{1/2} = \frac{Q_s}{|\eta|}.$$

Typical parametric dependence of the phase space amplitudes is

$$\hat{\delta} \sim \mathcal{A}^{1/2}V^{1/4}h^{1/4}|\eta|^{-1/4}\gamma^{-3/4}, \qquad \hat{\theta} \sim \mathcal{A}^{1/2}V^{-1/4}h^{-1/4}|\eta|^{1/4}\gamma^{-1/4},$$

For a beam with rms momentum and phase **spreads** σ_δ and σ_ϕ, the rms phase space area is $A_{rms}=\pi\sigma_\phi\sigma_\delta=\pi h\sigma_\theta\sigma_\delta$. Here θ is the orbital angle and ϕ is the rf phase angle. A beam bunch with Gaussian distribution is

$$\rho(\delta,\theta) = \frac{1}{2\pi\sigma_\delta\sigma_\theta}\exp\left\{-\frac{1}{2}\left[\frac{\theta^2}{\sigma_\theta{}^2}+\frac{\delta^2}{\sigma_\delta{}^2}\right]\right\}$$

Amplitude dependence of synchrotron tune:

$$H = \frac{1}{2}h\omega_0\eta\delta^2 + \frac{1}{2h\eta}\omega_0 Q_s^2\left[\varphi^2 - \frac{1}{3}\tan\phi_s\ \varphi^3 - \frac{1}{12}\ \varphi^4 + \cdots\right],$$

Carrying out canonical transformation with the generating function:

$$F_1(\phi,\psi) = -\frac{Q_s}{2h|\eta|}\varphi^2\tan\psi,$$

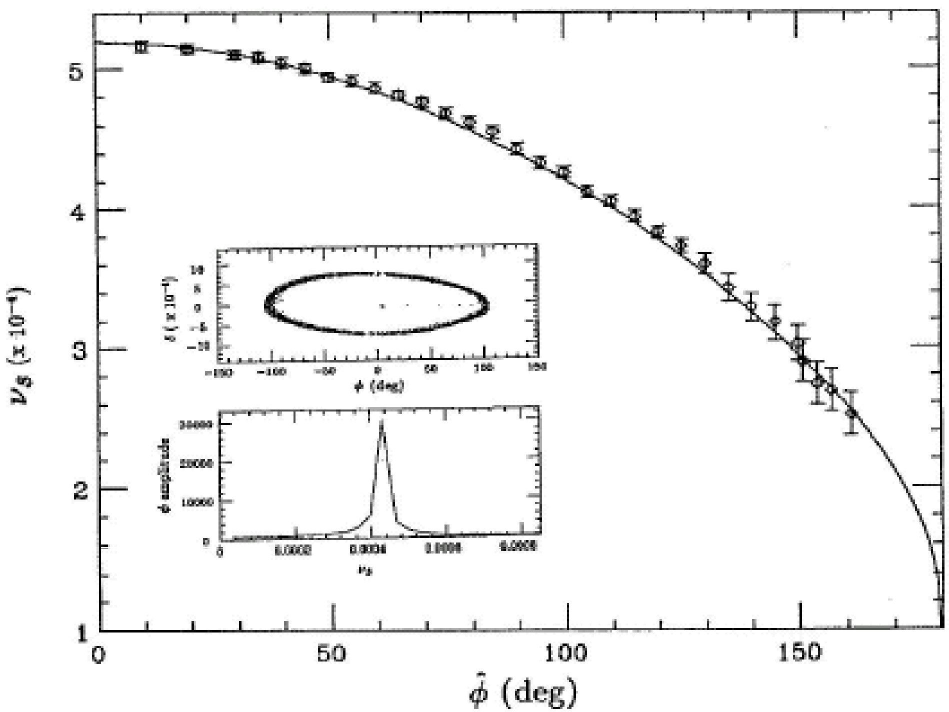
We obtain

$$\varphi = \sqrt{\frac{2h\eta J}{Q_s}}\cos\psi, \quad \delta = -\sqrt{\frac{2Q_s J}{h\eta}}\sin\psi,$$

$$H = \omega_0 Q_s J - \frac{\omega_0 h\eta}{16}\left(1 + \frac{5}{3}\tan^2\phi_s\right)J^2 + \cdots.$$

$$\tilde{Q}_s(J) \approx Q_s\left[1 - \frac{h\eta}{8Q_s}\left(1 + \frac{5}{3}\tan^2\phi_s\right)J\right].$$

$$Q_s(J) = Q_s\left(1 - \frac{1}{16}\left(1 + \frac{5}{3}\tan^2\phi_s\right)\hat{\varphi}^2\right)$$



Small amplitude motion around the unstable fixed point

$$\dot{\phi} = h\eta\omega_0\delta \qquad \dot{\delta} = \frac{\omega_0 e V_0}{2\pi\beta^2 E}(\sin\phi - \sin\phi_{\mathrm{s}})$$

Near the unstable fixed point we set ϕ=π–ϕs+φ. The equation of motion becomes

$$\dot{\delta} = \frac{\omega_0 e V_0}{2\pi\beta^2 E}(\sin(\pi - \phi_{\mathrm{s}} + \varphi) - \sin\phi_{\mathrm{s}}) \approx -\frac{\omega_0 e V_0 \cos\phi_{\mathrm{s}}}{2\pi\beta^2 E}\varphi$$

$$\dot{\varphi} = h\eta\omega_0\delta$$

Thus $$\ddot{\varphi} = -\frac{\omega_0^2 e V_0 h\eta\cos\phi_{\mathrm{s}}}{2\pi\beta^2 E}\varphi = \omega_{\mathrm{s}}^2\varphi, \quad \ddot{\delta} = \omega_{\mathrm{s}}^2\delta$$

The equation of motion around the UFP is hyperbolical.

$$\varphi = \varphi_0 \cosh\omega_{\mathrm{s}} t + \frac{h\eta\omega_0}{\omega_{\mathrm{s}}}\delta_0 \sinh\omega_{\mathrm{s}} t$$

$$\delta = \frac{\omega_{\mathrm{s}}}{h\eta\omega_0}\varphi_0 \sinh\omega_{\mathrm{s}} t + \delta_0 \cosh\omega_{\mathrm{s}} t$$

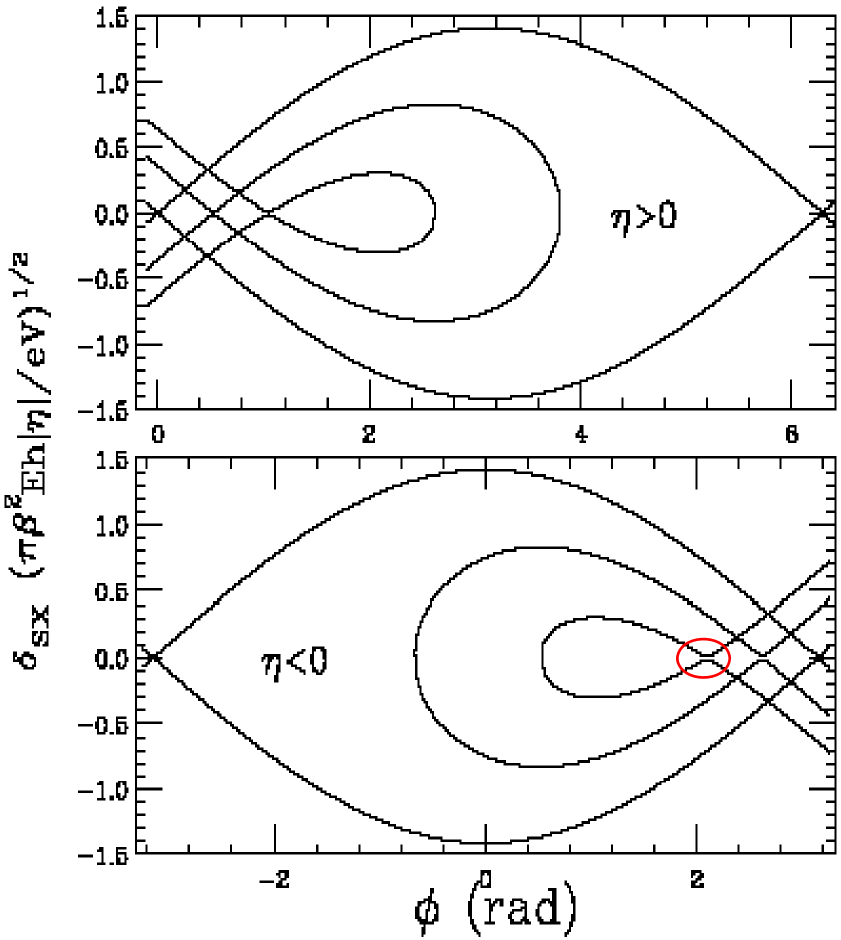
Define the normalized coordinates: $\tilde{\delta} = \frac{\delta}{\delta_0}, \quad \tilde{\varphi} = \frac{\varphi}{\varphi_0}$

The phase space ellipse becomes

$$\tilde{\delta}^2 - 2\left(\frac{\eta}{|\eta|}\tanh 2\omega_s t\right)\tilde{\delta}\tilde{\varphi} + \tilde{\varphi}^2 = \frac{1}{\cosh 2\omega_s t}$$

$$t \to \infty, \quad \tilde{\delta} \mp \tilde{\varphi} = 0$$

Note that the phase space ellipse becomes elongated, while the phase space area is preserved. The UFP can be used for bunch compression.



Requirement of rf systems:

1. Acceleration rate: $dE_0/dt=f_0eV\sin\phi_s$ with $V\sin\phi=2\pi R\rho(dB/dt)$
2. If $\eta<0$, $0<\phi_s<\pi/2$, if $\eta>0$, $\pi/2<\phi_s<\pi$.
3. The bucket area must be larger than the bunch area, f_{rf}, V, h.
4. The rf frequency is related to magnetic field by

$$\omega_{\rm rf} = h\frac{\beta c}{R_0} = \frac{he\rho B}{R_0\gamma m} = \frac{hc}{R_0}\left[\frac{B^2(t)}{B^2(t)+(mc^2/ec\rho)^2}\right]^{1/2}$$

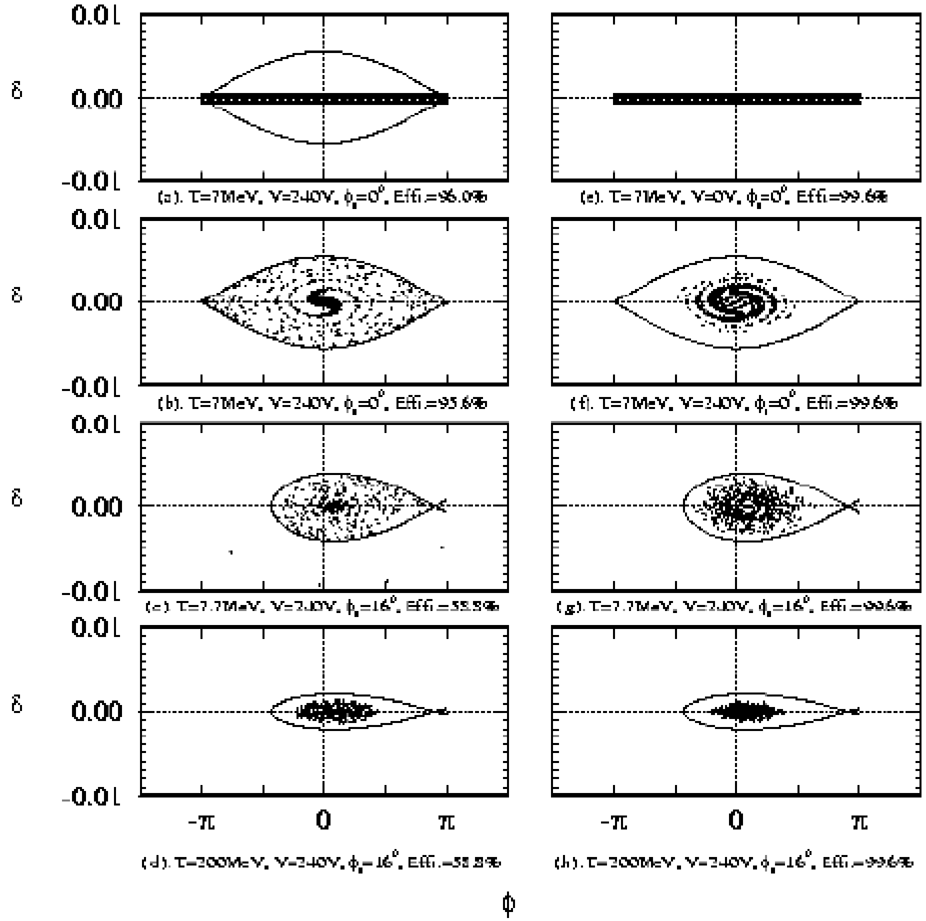
5. The bucket area and the bucket height are

	$(\phi, \frac{\Delta E}{\omega_0})$	(ϕ, δ)	$(\phi, \frac{h\lvert\eta\rvert}{\nu_s}\delta)$
Bucket Area	$16\left(\frac{\beta^2 EeV}{2\pi\omega_0^2 h\lvert\eta\rvert}\right)^{1/2}\alpha_b(\phi_s)$	$16\left(\frac{eV}{2\pi\beta^2 Eh\lvert\eta\rvert}\right)^{1/2}\alpha_b(\phi_s)$	$16\,\alpha_b(\phi_s)$
Bucket Height	$2\left(\frac{\beta^2 EeV}{2\pi\omega_0^2 h\lvert\eta\rvert}\right)^{1/2}Y(\phi_s)$	$2\left(\frac{eV}{2\pi\beta^2 Eh\lvert\eta\rvert}\right)^{1/2}Y(\phi_s)$	$2\,Y(\phi_s)$

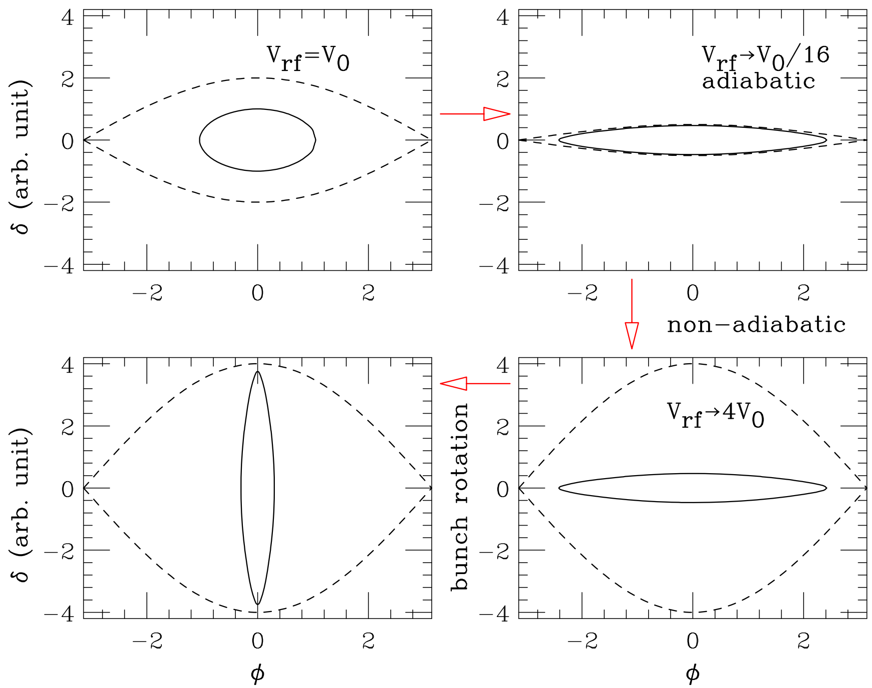
$$Q_s = \sqrt{heV\lvert\eta_0\cos\phi_s\rvert/2\pi\beta^2 E}$$

	AGS	RHIC	FNAL-MI	FNAL-BST	SSC	Cooler
$\nu_s(\times 10^{-3})$	32.3	0.589	13.7	18.1	0.839	0.395

RF beam manipulations:
(1) adiabatic capture



(2) Bunch compression



Homework#4

In proton accelerators, the rf gymnastics for bunch rotation is performed by adiabatically lowering the voltage from V_1 to V_2 and suddenly raising the voltage from V_2 to V_1 (see also Exercise 3.2.5). Using Eq. (3.51) and conservation of phase-space area, show that the bunch length in the final step is

$$\hat{\theta}_{\rm final} = \left(\frac{\nu_{s2}}{\nu_{s1}}\right)^{1/2} \hat{\theta}_{\rm initial},$$

where $\hat{\theta}_{\rm initial}$ is the initial bunch length in orbital angle variable, and ν_{s1} and ν_{s2} are the synchrotron tune at voltages V_1 and V_2. Apply the bunch rotation scheme to proton beams at $E = 120$ GeV in the Fermilab Main Injector, where the circumference is 3319.4 m, the harmonic number is $h = 588$, the transition energy is $\gamma_T = 21.8$, and the phase-space area is $\mathcal{A} = 0.05$ eV-s for 6×10^{10} protons. Find the voltage V_2 such that the final bunch length is 0.15 ns with an initial voltage $V_1 = 4$ MV. The energy of the secondary antiprotons is 8.9 GeV. If the acceptance of the antiproton beam is $\pm 3\%$, what is the phase-space area of the antiproton beams? If the antiproton production efficiency is 10^{-5}, what is the phase-space density of the antiproton beams?

$$\hat{\delta} = \mathcal{A}^{1/2}\left(\frac{\omega_0}{\pi\beta^2 E}\right)^{1/2}\left(\frac{heV|\cos\phi_s|}{2\pi\beta^2 E|\eta|}\right)^{1/4},$$

$$\hat{\theta} = \frac{1}{h}\hat{\phi} = \mathcal{A}^{1/2}\left(\frac{\omega_0}{\pi\beta^2 E}\right)^{1/2}\left(\frac{2\pi\beta^2 E|\eta|}{heV|\cos\phi_s|}\right)^{1/4}, \tag{3.51}$$

$$\frac{\hat{\delta}}{\hat{\theta}} = \left(\frac{heV|\cos\phi_s|}{2\pi\beta^2 E|\eta|}\right)^{1/2} = \frac{Q_s}{|\eta|}. \tag{3.52}$$

Solution HW#4

In the bunch rotation manipulation, the evolution of the beam bunch is

$$(V_1, \delta_0, \theta_0) \xrightarrow{\text{adiabatic}} (V_2, \delta_1, \theta_1) \xrightarrow{\text{non-adiabatic}} (V_1, \delta_1, \theta_1) \xrightarrow{\text{rotate}} (V_1, \delta_2, \theta_2)$$

The phase-space area enclosed by the ellipse is invariant in linear synchrotron motion, i.e.

$$\pi\delta_0\theta_0 = \pi\delta_1\theta_1$$

From the relations

$$\delta_0 = \frac{\nu_{s2}}{\eta}\theta_0 \qquad \delta_1 = \frac{\nu_{s2}}{\eta}\theta_1$$

we get $\theta_1^2\nu_{s2} = \theta_0^2\nu_{s1}$. From (3.57), the rotation in the last step has the relations

$$\delta_2 = \frac{\nu_{s1}}{\eta}\theta_1 \qquad \theta_2 = \frac{\eta}{\nu_{s1}}\delta_1$$

The final bunch length is

$$\theta_2 = \frac{\eta}{\nu_{s1}}\delta_1 = \frac{\nu_{s2}}{\nu_{s1}}\theta_1 = \left(\frac{\nu_{s2}}{\nu_{s1}}\right)^{\frac{1}{2}}\theta_0 = \left(\frac{V_2}{V_1}\right)^{\frac{1}{4}}\theta_0$$

Thus

$\omega_0 = 0.5679$ MHz, $\quad \theta_2 = 8.518 \times 10^{-5}, \quad \theta_0 = 2.487 \times 10^{-4}, \quad V_2 = 469.2$ (keV)

Double rf system:

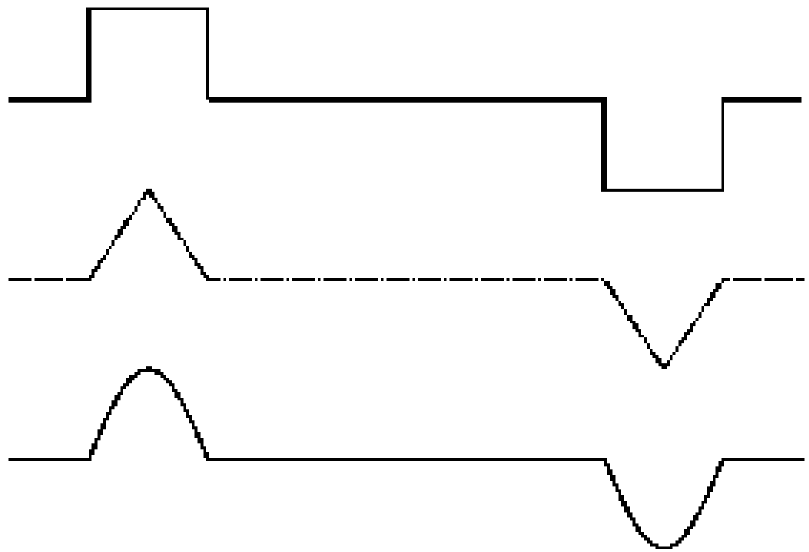
$$\dot{\delta} = \frac{\omega_0 e V_1}{2\pi\beta^2 E}\left\{\sin\phi - \sin\phi_{1s} + \frac{V_2}{V_1}\left(\sin\left[\phi_{2s} + \frac{h_2}{h_1}(\phi - \phi_{1s})\right] - \sin\phi_{2s}\right)\right\}$$

$$H = \frac{1}{2}\nu_s \mathcal{P}^2 + V(\phi),$$

$$\begin{aligned} V(\phi) \quad = \quad & \nu_s\{(\cos\phi_{1s} - \cos\phi) + (\phi_{1s} - \phi)\sin\phi_{1s} \\ & -\frac{r}{h}\left[\cos\phi_{2s} - \cos\left(\phi_{2s} + h(\phi - \phi_{1s})\right) - h(\phi - \phi_{1s})\sin\phi_{2s}\right]\} \end{aligned}$$

With double rf system, the beam bunch can have a larger tune spread.

Barrier buckets: The sinusoidal rf potential can provide phase focusing for particle motion. In fact, the barrier bucket of any potential shape can also provide stability of particle motion. The barrier bucket can also be used for bunch beam manipulation.



Homework#5

The equilibrium distribution in linearized synchrotron phase space is a function of the invariant ellipse with $\sigma_\theta = |\eta|\sigma_\delta/\nu_s$. When a mismatched Gaussian beam

$$\rho(\delta, \theta) = \frac{N_B e}{2\pi\sigma_\delta\sigma_\theta} \exp\left\{-\frac{1}{2}\left[\frac{\theta^2}{\sigma_\theta{}^2} + \frac{\delta^2}{\sigma_\delta{}^2}\right]\right\}$$

is injected into the synchrotron at time t = 0, what is the time evolution of the beam?

Show that the projection of the beam distribution function onto the θ axis is

$$\rho(\theta, t) = \frac{N_B e}{\sqrt{2\pi}\tilde{\sigma}} e^{-\theta^2/2\tilde{\sigma}^2} \qquad \tilde{\sigma}^2 = \sigma_\theta^2 \cos^2 \omega_s t + (|\eta|\sigma_\delta/\nu_s)^2 \sin^2 \omega_s t.$$

Show that the peak current is $\hat{I}(t) = N_B e\omega_0/\sqrt{2\pi}\tilde{\sigma}$.

Solution HW#5

Define $x = \theta$, and $p = |\eta|\delta/\nu_s$ as the normalized phase space coordinates. The tori of linearized synchrotron motion in the normalized phase-space are circles with $x^2 + p^2 =$ constant. The initial rms beam widths in the normalized phase-space coordinates becomes $\sigma_{x0} = \sigma_\theta$ and $\sigma_{p0} = |\eta|\sigma_\delta/\nu_s$. Let $(\sigma_{x0}/\sigma_{p0})^2 = 1 + X$. Then the initial injected beam distribution is

$$\rho(x_0, p_0) = \frac{N_B e}{2\pi\sigma_{x0}\sigma_{p0}} \exp\left\{-\frac{1}{2\sigma_{x0}^2}\left[x_0^2 + (1+X)p_0^2\right]\right\}$$

The linear synchrotron motion can be expressed as

$$x = x_0 \cos\omega_s t + p_0 \sin\omega_s t$$

$$p = -x_0 \sin\omega_s t + p_0 \cos\omega_s t.$$

$$\frac{1}{2\sigma_{x0}^2}[x_0^2 + (1+X)p_0^2] = \frac{x^2}{2\sigma_x^2(t)} + \frac{1 + X\cos^2\omega_s t}{2\sigma_{x0}^2}\left[p + \frac{X\sin\omega_s t\cos\omega_s t}{1 + X\cos^2\omega_s t}x\right]^2$$

$$\sigma_x^2(t) = \sigma_{x0}^2\cos^2\omega_s t + \sigma_{p0}^2\sin^2\omega_s t$$

The beam distribution ρ(θ) can be obtained by integrating over the coordinate

$$\rho(x) = \int \rho(x,p)\,dx\,dp = \frac{N_B e}{\sqrt{2\pi}\sigma_x(t)} e^{-\frac{x^2}{2\sigma_x^2(t)}}.$$

Summary

1. Longitudinal rf electric field in accelerator causes charged particles to bunch into groups. The stable phase area of the phase and off-momentum coordinates is called the **bucket area**, and the torus that passes through the UFP is called the separatrix. The phase space area of the beam is c
2. Particles in the bucket execute synchrotron motion, where the number of oscillations per revolution is called synchrotron tune. Typical synchrotron tune in synchrotrons is about 0.001 to 0.1.
3. Longitudinal rf electric field can be used to manipulate the beam profiles by trading the bunch length vs momentum spread. Similarly the transverse rf electromagnetic field can be used to co-relate the longitudinal and transverse coordinates.

Nonlinear beam dynamics in betatron and synchrotron motion

$$\tilde{H} \approx -p\left(1+\frac{x}{\rho}\right)+\frac{1+x/\rho}{2p}\left[(p_x-eA_x)^2+(p_z-eA_z)^2\right]-eA_s.$$

$$H=\frac{1}{2}x'^2+\frac{1}{2}K_x(s)x^2+\frac{1}{2}z'^2+\frac{1}{2}K_z(s)z^2-eA_{s,NL}(x,z,s)$$

$$A_s=B_0\Re\left[\sum_{n=0}^{\infty}\frac{b_n+ja_n}{n+1}(x+jz)^{n+1}\right]$$

$$A_{s,NL}=\frac{B_0b_3}{3}\left(x^3-3xz\right)+\frac{B_0b_4}{4}\left(x^4-6x^2z^2+z^4\right)+\ldots$$

$$H = \boxed{\frac{1}{2}x'^2 + \frac{1}{2}K_x(s)x^2 + \frac{1}{2}z'^2 + \frac{1}{2}K_z(s)z^2} - eA_{s,NL}(x,z,s)$$

$$y'' + K(s)y = 0, \qquad H = \frac{1}{2}y'^2 + \frac{1}{2}K(s)y^2,$$

$$y = \sqrt{2\beta J}\ \cos\psi, \qquad y' = -\sqrt{\frac{2J}{\beta}}\ [\sin\psi + \alpha\cos\psi],$$

$$F_1(y,\psi) = \int_0^y y'dy = -\frac{y^2}{2\beta}(\tan\psi - \frac{\beta'}{2}), \qquad J = -\frac{\partial F_1}{\partial\psi} = \frac{y^2}{2\beta}\sec^2\psi = \frac{1}{2\beta}[y^2 + (\beta y' + \alpha y)^2]$$

$$\tilde{H} = H + \frac{\partial F_1}{\partial s} = \frac{J}{\beta}. \qquad \frac{dJ}{ds} = -\frac{\partial\tilde{H}}{\partial\psi} = 0. \qquad \psi' = \partial\tilde{H}/\partial J = 1/\beta(s),$$

Note that the Hamiltonian Ĥ is s-dependent, and the Hamiltonian has different value at different locations. We can remove this flutter of Hamiltonian by making a Canonical transformation, and by employing the orbital angle θ=s/R as the independent coordinate.

$$F_2(\psi,\bar{J}) = \left(\psi - \int_0^s \frac{ds}{\beta} + \nu\theta\right)\bar{J}$$

$$F_2(\psi, \bar{J}) = \left(\psi - \int_0^s \frac{ds}{\beta} + \nu\theta\right)\bar{J}$$

The conjugate phase space coordinates are $\bar{\psi} = \psi - \int_0^s \frac{ds}{\beta} + \nu\theta, \quad \bar{J} = J,$

Thus the new Hamiltonian becomes $\overline{H} = R\left\{\widetilde{H} + \frac{\partial F_2}{\partial s}\right\} = R\left\{\frac{J}{\beta} + \frac{\nu}{R}J - \frac{1}{\beta}J\right\} = \nu J$

$$\frac{d\overline{\psi}}{d\theta} = Q = \frac{\partial \overline{H}}{\partial J} = \nu, \quad \frac{dJ}{d\theta} = -\frac{\partial \overline{H}}{\partial \overline{\psi}} = 0$$

$$y = \sqrt{2\beta\bar{J}}\cos\left(\bar{\psi} + \chi(s) - \nu\theta\right),$$
$$\mathcal{P}_y = \beta y' + \alpha y = -\sqrt{2\beta\bar{J}}\sin(\bar{\psi} + \chi(s) - \nu\theta),$$

$$\chi(s) = \int_0^s \frac{ds}{\beta}$$

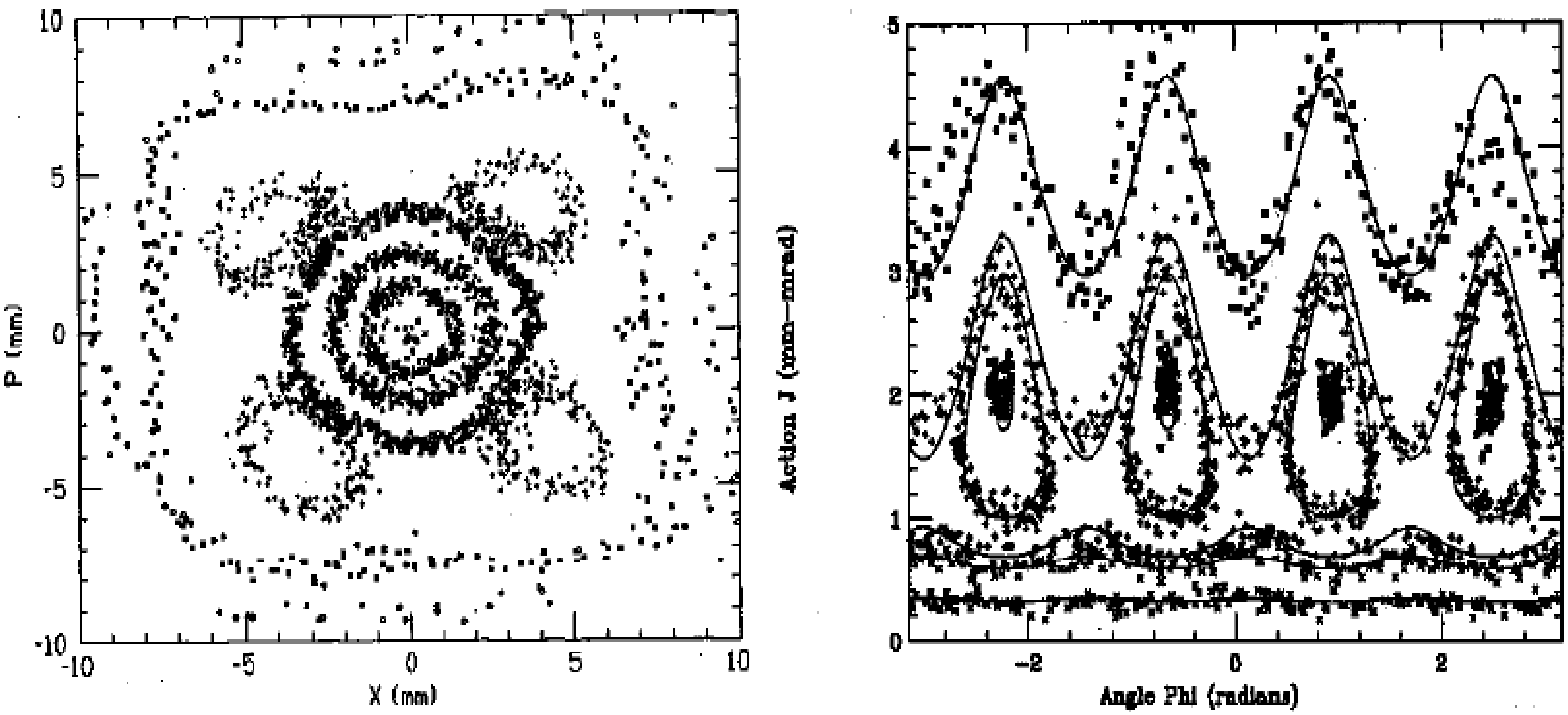
$$H = \frac{1}{2}x'^2 + \frac{1}{2}K_x(s)x^2 + \frac{1}{2}z'^2 + \frac{1}{2}K_z(s)z^2 - eA_{s,NL}(x, z, s)$$

$$\widetilde{H} = \nu_x J_x + \nu_z J_z - eA_{s,NL}(J_x, \psi_x, J_x, \psi_x, \theta)$$

Near a fourth-order 1D resonance, the Hamiltonian can be approximated by

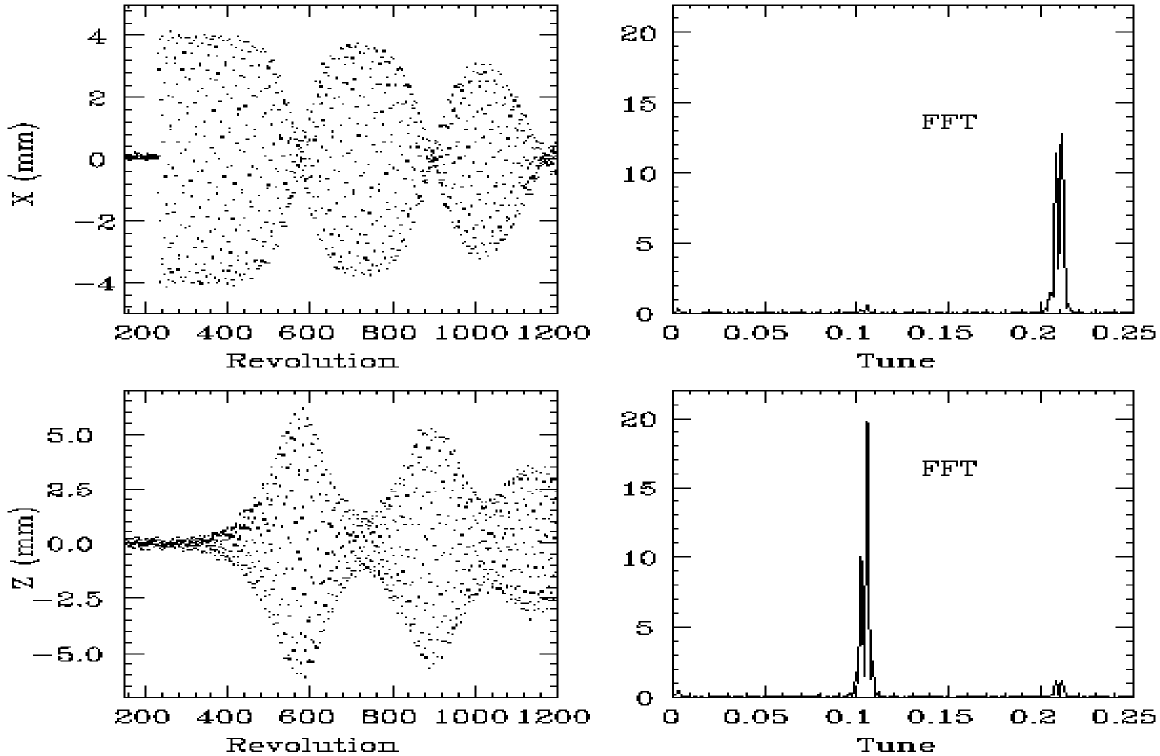
$$H = \nu_x J_x + \frac{1}{2}\alpha_{xx} J_x^2 + G_{4,0,\ell} J_x^2 \cos(4\psi_x - \ell\theta + \chi), \tag{2.387}$$

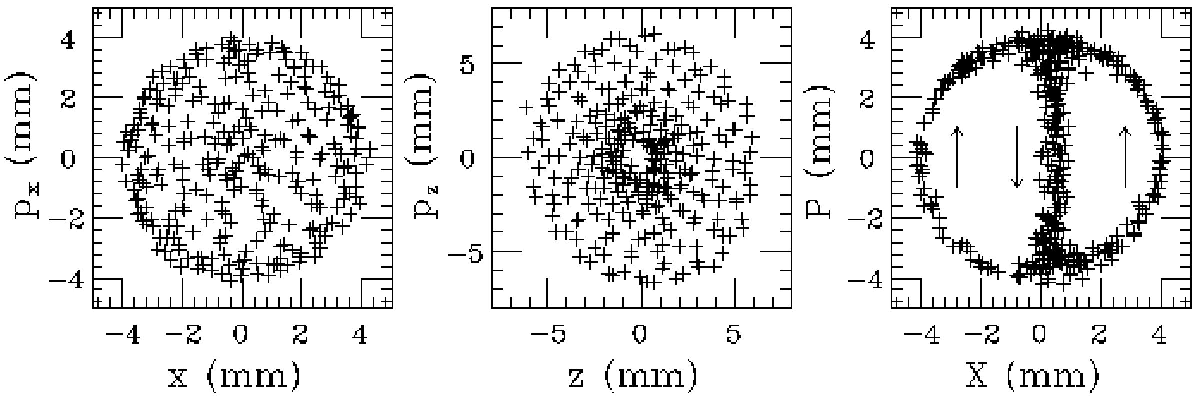
where the resonance strength $G_{4,0,\ell}$ can be obtained from the Fourier transformation of the effective particle Hamiltonian in the synchrotron. The Poincaré map near a fourth-order resonance $4\nu_x = 15$ measured at the IUCF cooler ring is shown in Fig. 2.54, where the left plot shows the Poincaré map in the normalized (x, P_x) phase space and the right plot shows the Poincaré map in action-angle variables.



$\nu_x - 2\,\nu_z = \ell$

$$H_1 = Q_x J_x + Q_z J_z + \frac{\alpha_{xx}}{2} J_x^2 + \alpha_{xz} J_x J_z + \frac{\alpha_{zz}}{2} J_z^2$$
$$+ 2^{3/2} B J_x^{1/2} J_z \cos(\phi_x - 2\phi_z + 6\theta + \mu),$$





The betatron phase space can be visualized as a space filled by invariant tori, even near a nonlinear resonance. For a difference resonance, the invariant is bounded!

- Take $2\nu_x+2\nu_z$ resonance as an example, we expect to see particle loss through tori as shown in the graph below. This means that the betatron phase space is filled with resonance lines, where particles that locked onto a resonance will leak out to a large amplitude betatron motion through these resonance tori. The invariant tori are unbounded for sum resonances!
- Experiments has yet to be carried out!

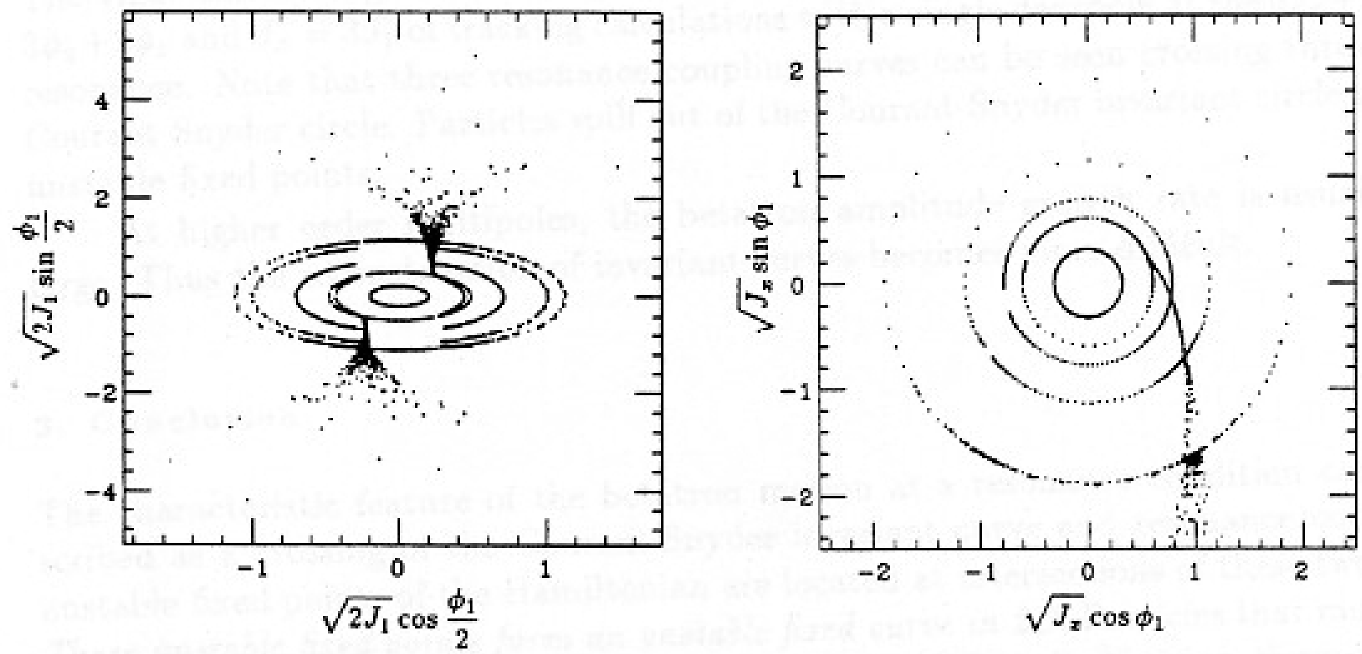
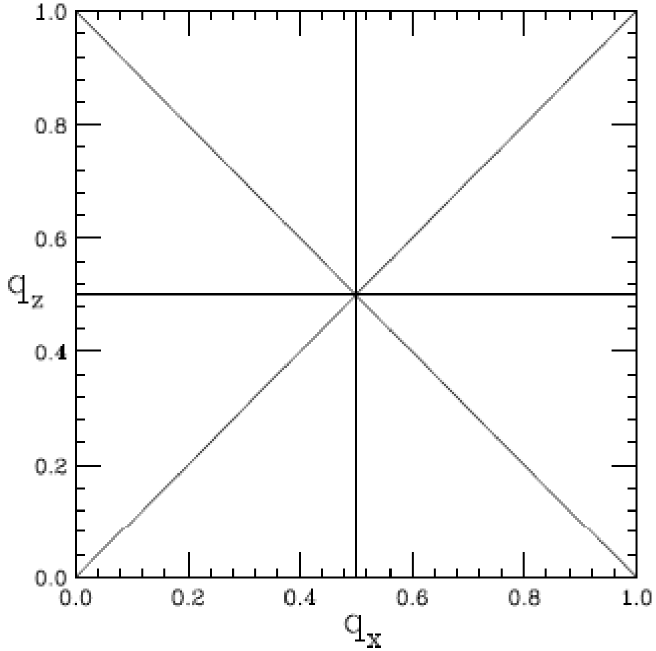
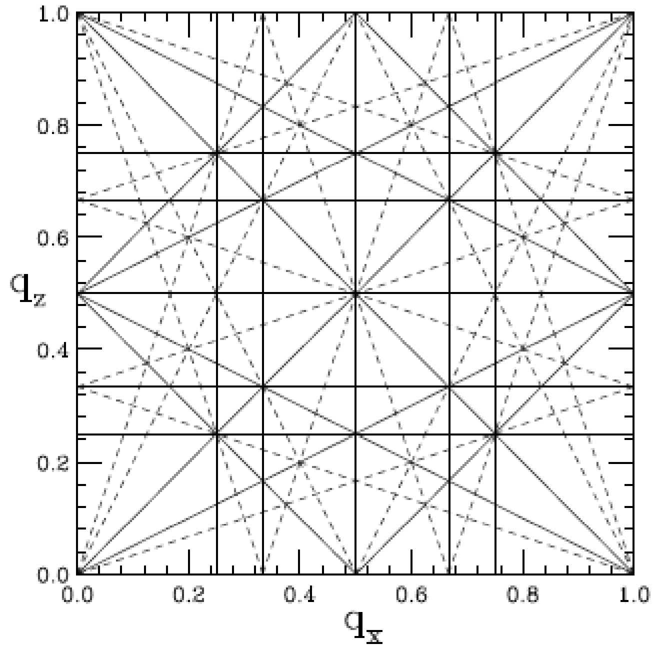


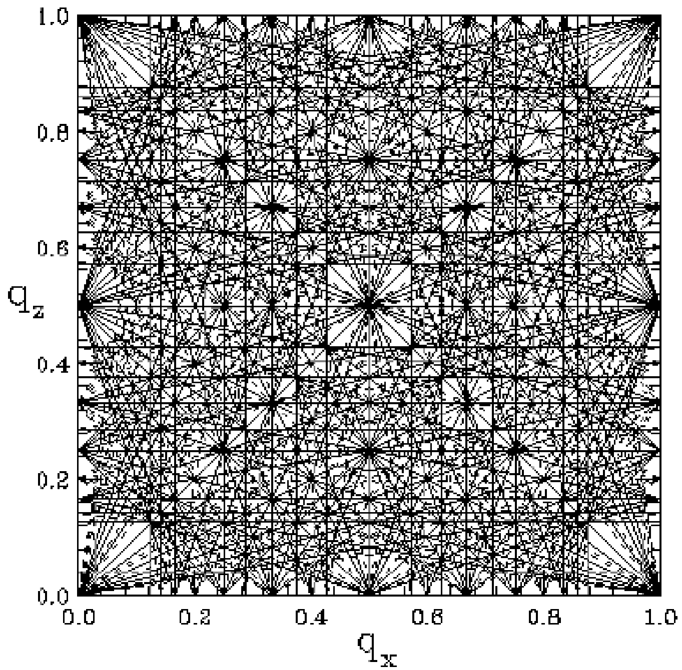
Figure 3. The Poincaré maps (see text for explanation) are shown for a Simple tracking calculation with a single octupole at a $2\nu_x+2\nu_z=\ell$ resonance.

Linear resonances





Resonances up to 4th order

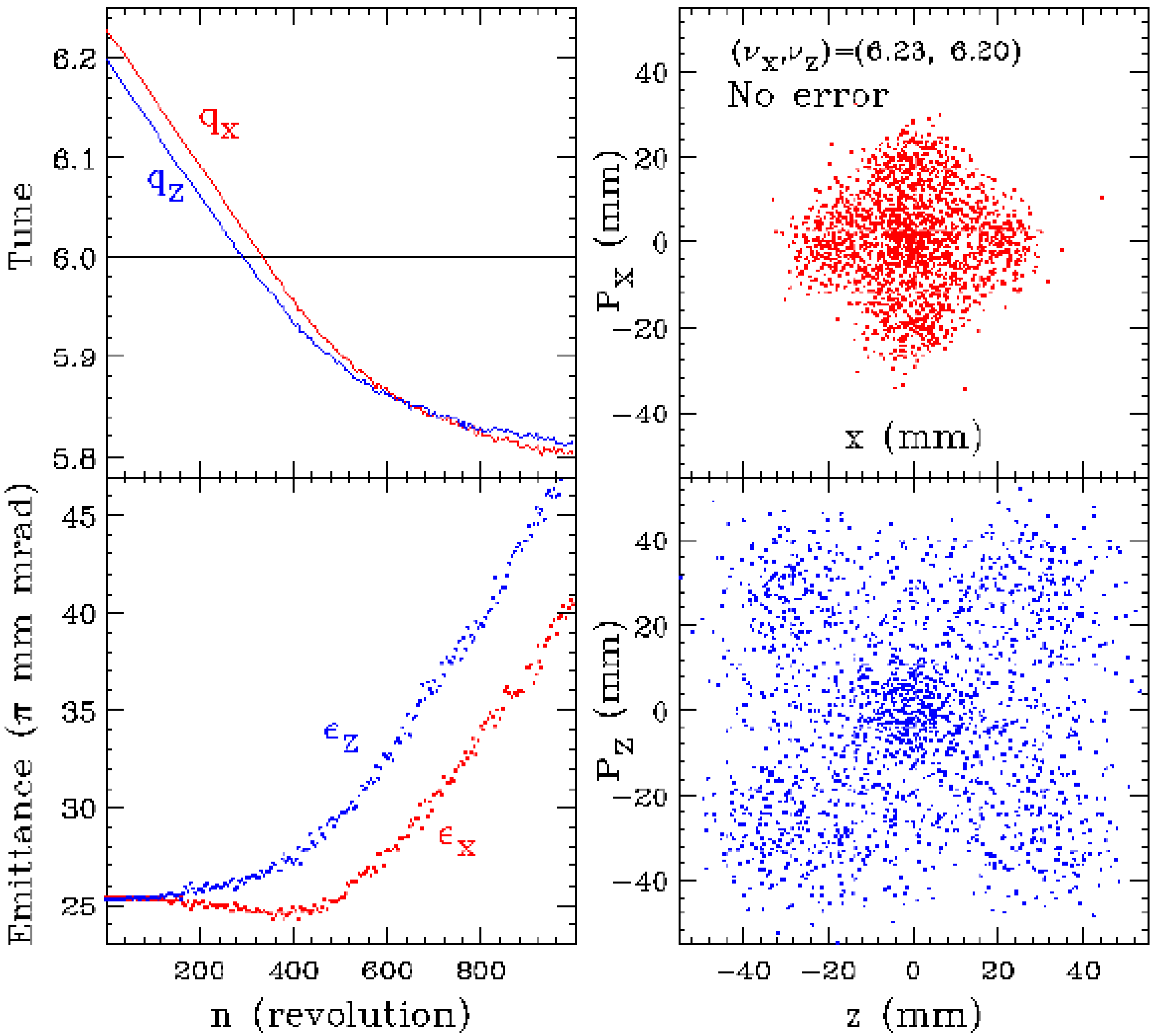


Up to 8th order resonances

Consider an accelerator with 24 superperiods, and set the bare betatron tunes just above the 4th order **systematic space charge resonance**!

For example: choose the bare tunes at (6.23, 6.20), and 1000 injection-turns, with a total tune-shift of 1.1, what will be the final beam distribution?

Phase space map at the end of injection.



The tune for small amplitude particles continue to decrease as the particle is injected!

What is the effect of the nonlinear systematic space charge resonances on beam emittances?

The space charge potential has the form of $\exp(-(x^2+z^2)/4\sigma^2)$. We know that the Montque resonance is produced by the x^2z^2 term in the potential. How about the systematic resonance induced by the terms x^4, z^4, x^2z^2, x^6, x^4z^2, x^2z^4, z^6, etc? Since the space charge potential follows the beam profile, which has the same superperiodicity, systematic resonances are located at $4q_x=P$, $4q_z=P$, $2q_x+2q_z=P$, $6q_x=P$, $6q_z=P$, etc.

What is the effects of systematic resonances?

1) S.Y. Lee, PRL97, 104801 (2005); NJP 08, 291
2) X Pang, HB08, 118 (2008)
3) S. Machida, NIMA 384, 316 (1997)
4) Ingo Hofmann, Giuliano Franchetti, and Alexei V. Fedotov , HB2002, AIP conference proceedings
5) S. Igarashi et al., observed at the KEKPS at injection, PAC2003, p.2610 (2003)
6) Oliver Boine-Frankenheim observed the 4th order resonance in bunch rotation at the SIS18 simulations.

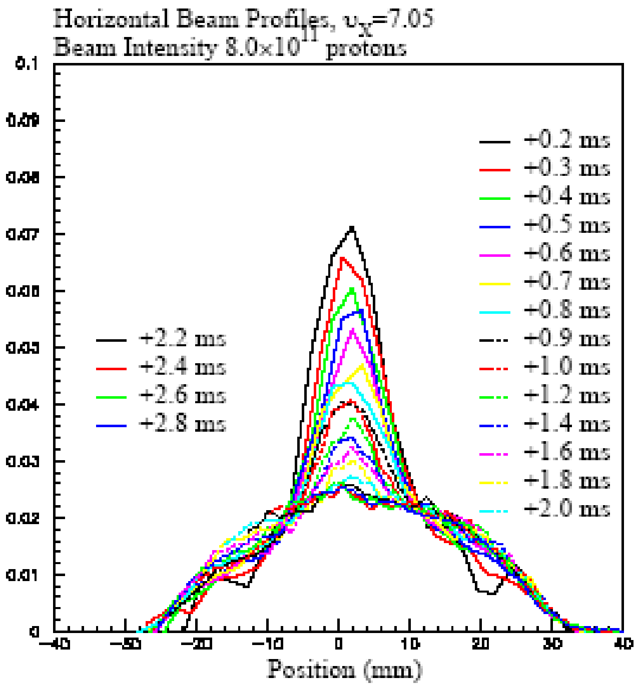
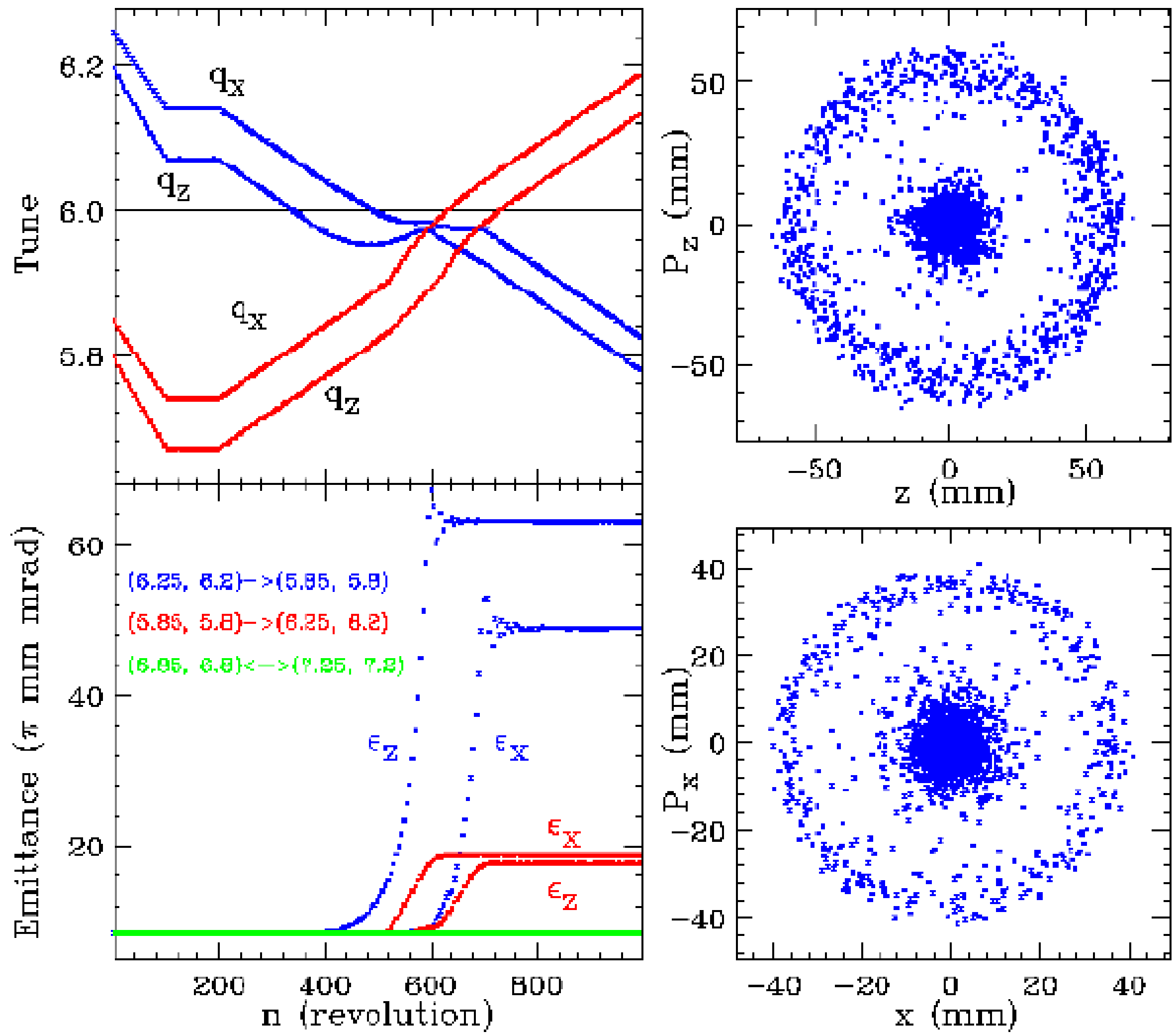
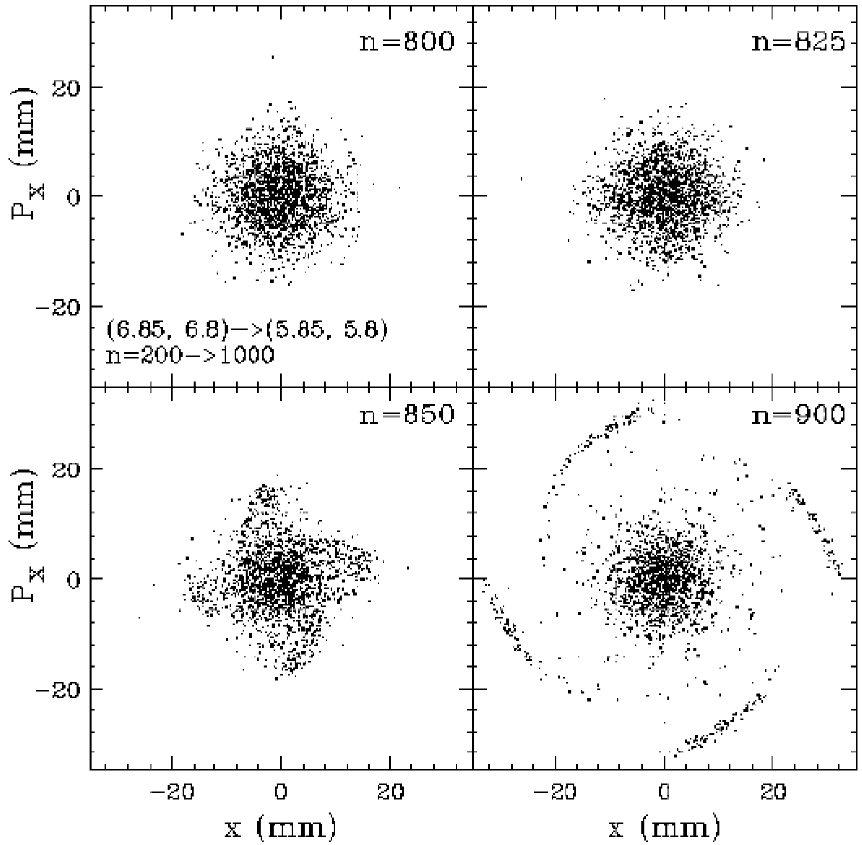


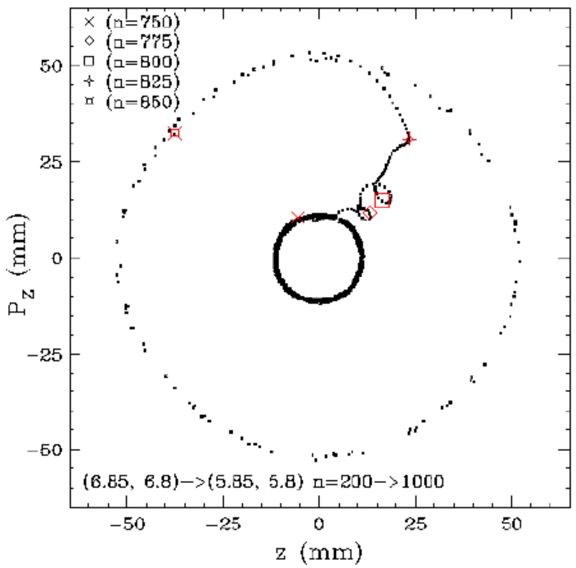
Figure 1: Horizontal beam profiles 0.2 ~ 2.8 ms after injection when the horizontal tune was 7.05 and the injection beam intensity was 8.0×10^{11} protons.

What happens if the betatron tunes are ramped through the systematic space charge resonance? Emittance growth of **nonscaling** FFAG in crossing systematic space charge resonance:

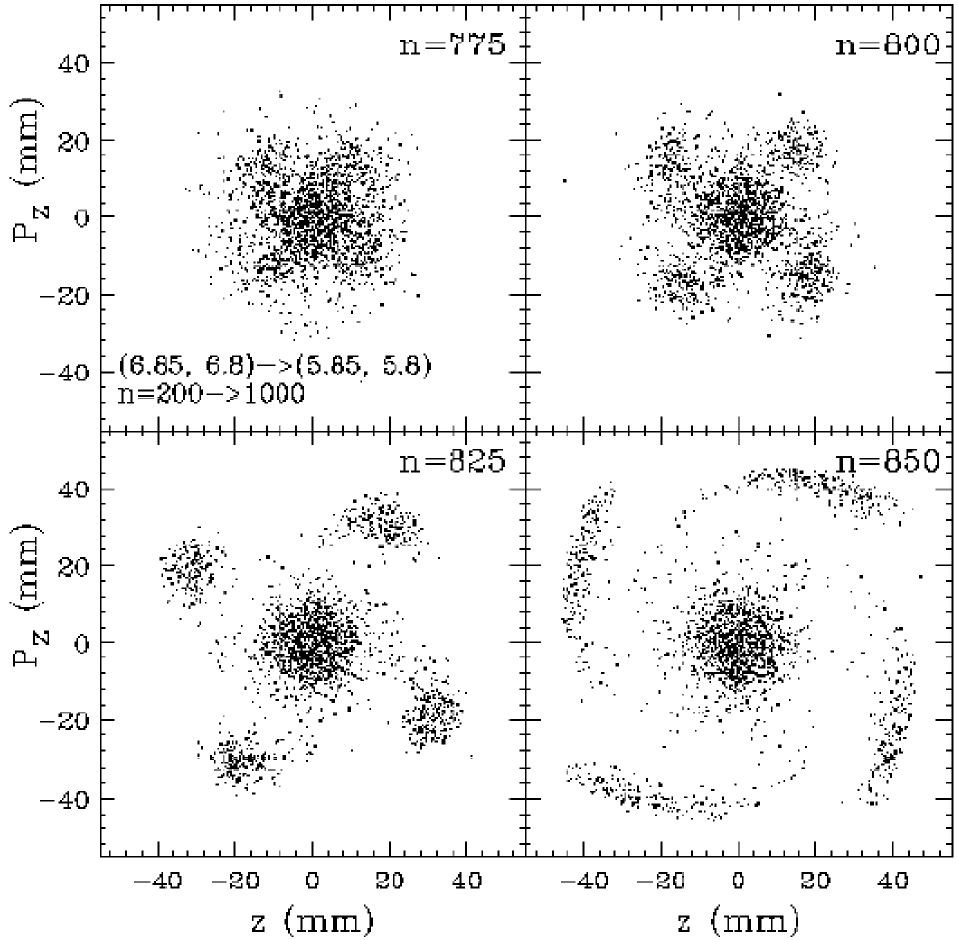




n=800
n=825
n=850
n=900
(6.85, 6.8)->(5.85, 5.8)
n=200->1000
P_x (mm)
x (mm)



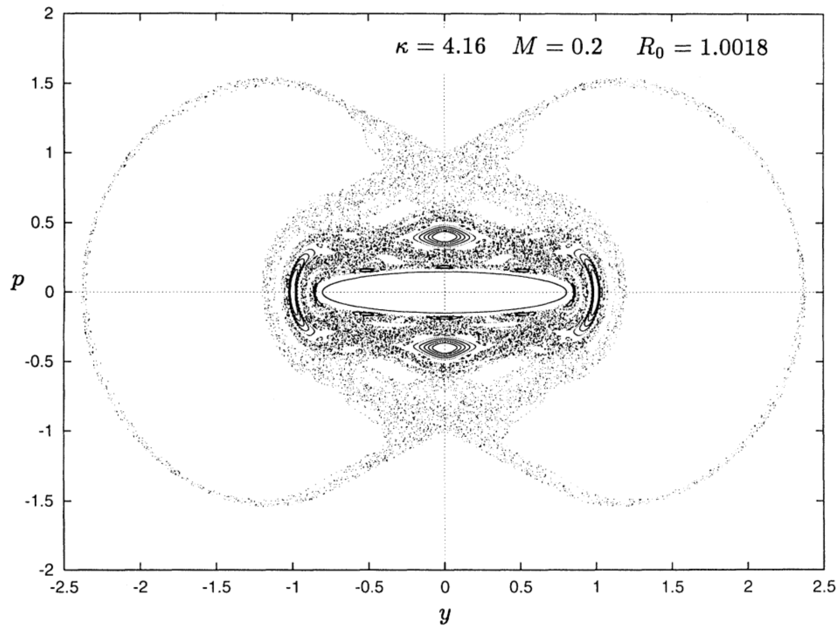
× (n=750)
◇ (n=775)
□ (n=800)
+ (n=825)
(n=850)
(6.85, 6.8)->(5.85, 5.8) n=200->1000
P_z (mm)
z (mm)



n=775
n=800
n=825
n=850
(6.85, 6.8)->(5.85, 5.8)
n=200->1000
P_z (mm)
z (mm)

Other emittance dilution mechanisms

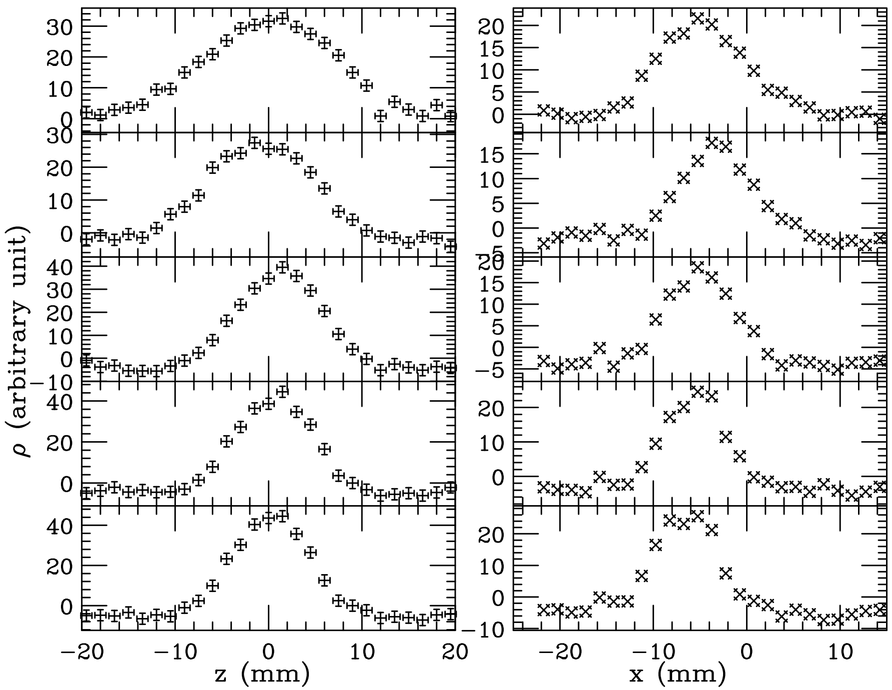
1. Since 1950, we believe that the **half-integer stopband** (or the envelope resonance by Sacherer) is the main cause of emittance growth in low energy high energy accelerators. This mechanism is also related to halo formation of high intensity linac due to 2:1 resonance. Phys. Rev. E 51, 1609 (1995), ibid 3529 (1995).
2. Space charge Montague resonance ($2\nu_x - 2\nu_z = 0$ or P)
3. Other resonances …



At Fermilab booster, there is an IPM for turn-by-turn beam profile measurement by averaging 53 bunches.

Data analysis of IPM data: The injection-turn numbers at Fermilab Booster were varied from 2 to 18. The gate of the ionization profile monitor was about 1s, or the measured profile was the average of about 52 bunches. The experimental condition for all data sets was the same as the normal operational condition with the corrector package, e.g., trim quadrupole, etc. The profile data at each revolution is fitted with a Gaussian plus polynomial model:

$$p(y) = a + by + A\exp\left(-\frac{(y-y_0)^2}{2\sigma_y^2}\right),$$



For the vertical plane, we can obtain emittance by

$$\epsilon_{y,\mathrm{rms}} = \frac{\sigma_y^2}{\beta_y},$$

Use basic accelerator physics information to extract the horizontal emittance!

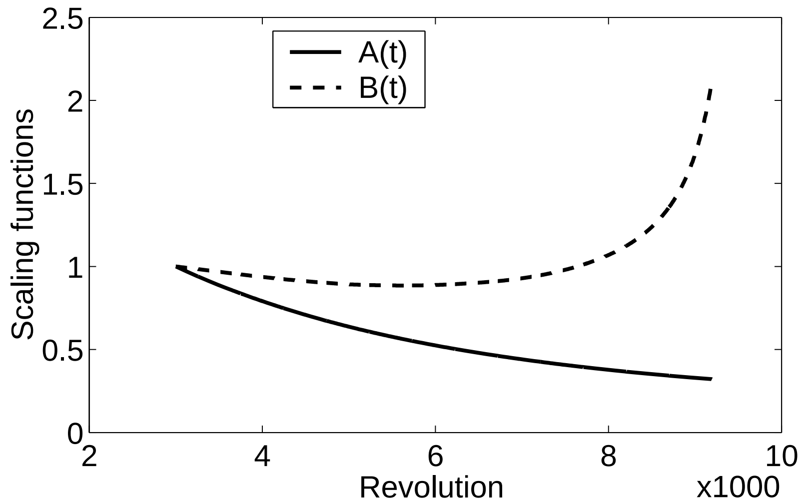
Assuming that the horizontal transverse phase-space distribution is uncorrelated to the longitudinal phase-space distribution, we obtain

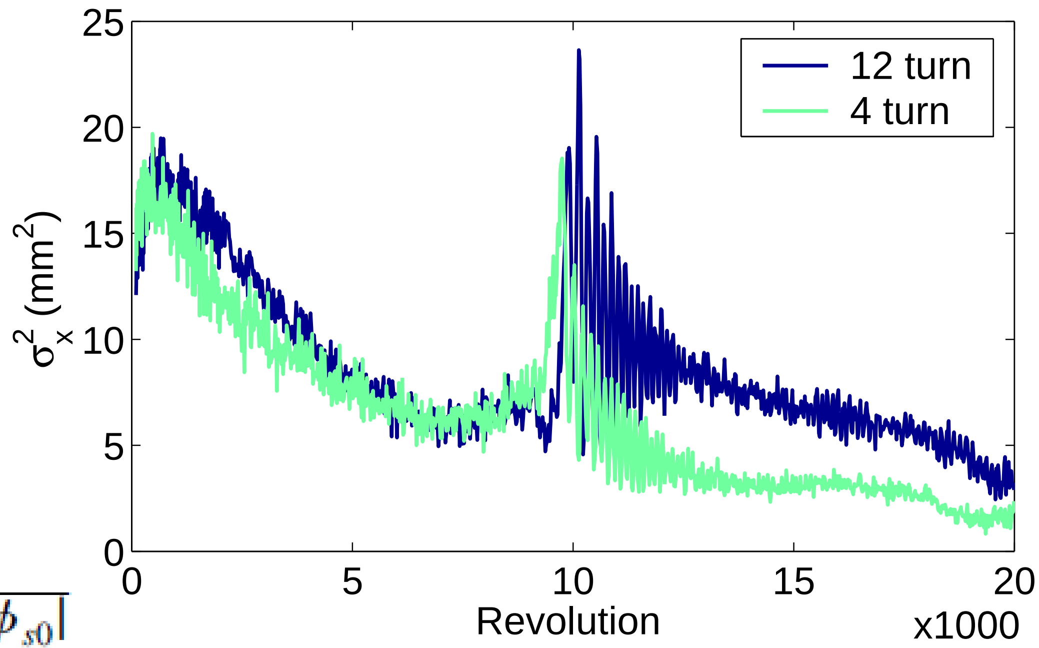
$$\sigma_x^2 = \beta_x \epsilon_{\rm rms} + D^2\sigma_\delta^2 = aA(t) + bB(t)$$

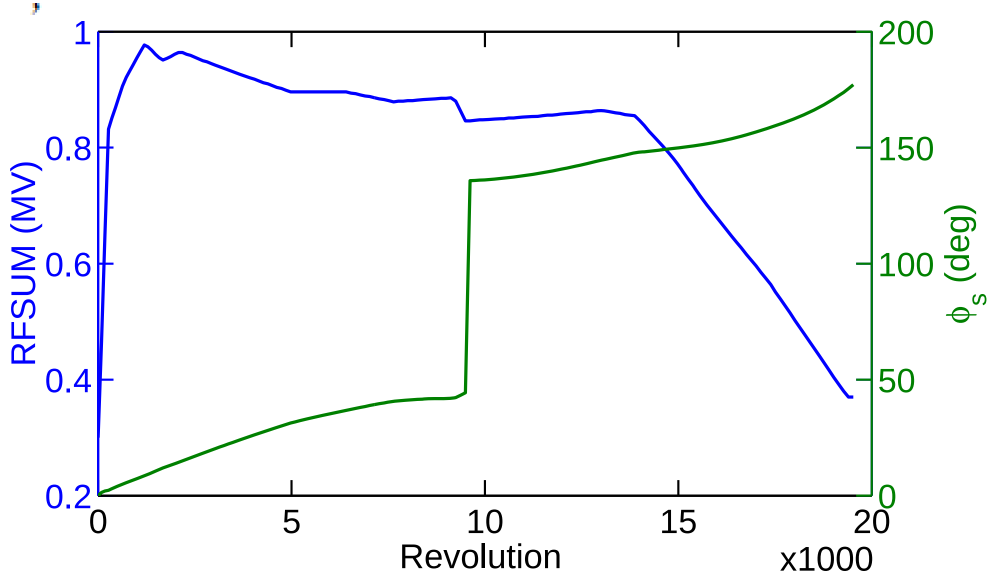
$$a = \epsilon_{\rm rms}^n \frac{\beta_{x0}}{\beta_0\gamma_0}, \qquad A(t) = \frac{\beta_x}{\beta_{x0}}\frac{\beta_0\gamma_0}{\beta\gamma},$$

$$b = D^2\sigma_{\delta 0}^2, \quad B(t) = \frac{\gamma_0\sqrt{\gamma_0|\eta_0|/V_0|\cos\phi_{s0}|}}{\gamma\sqrt{\gamma|\eta|/V|\cos\phi_s|}},$$

$$\sigma_\delta \sim V^{1/4}|\eta|^{-1/4}\gamma^{-3/4}$$



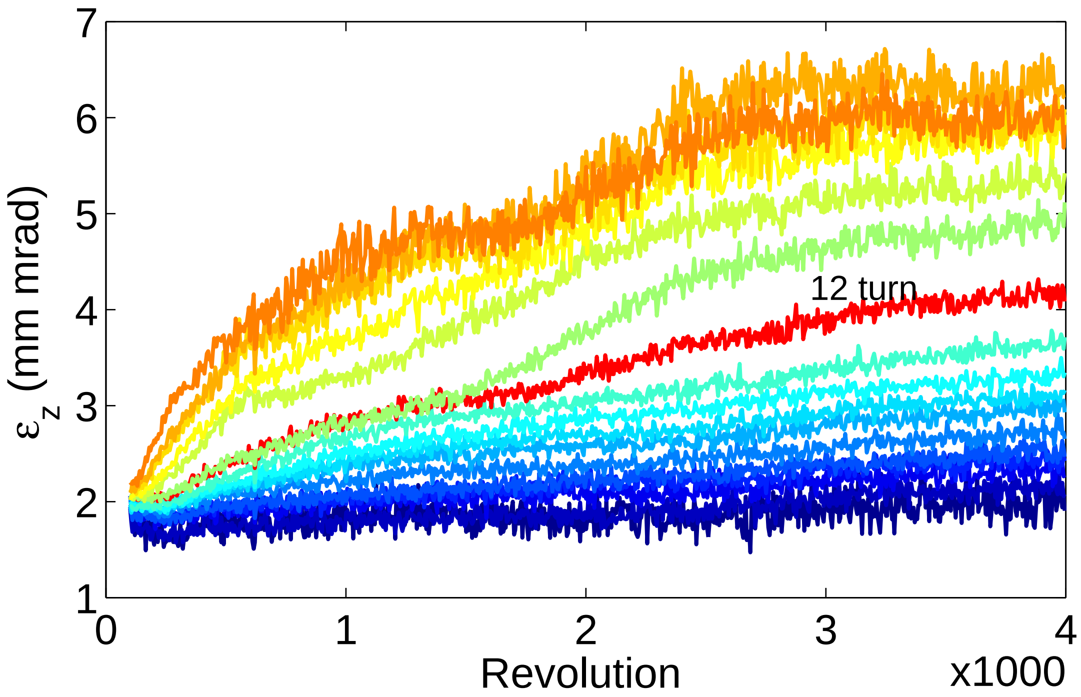




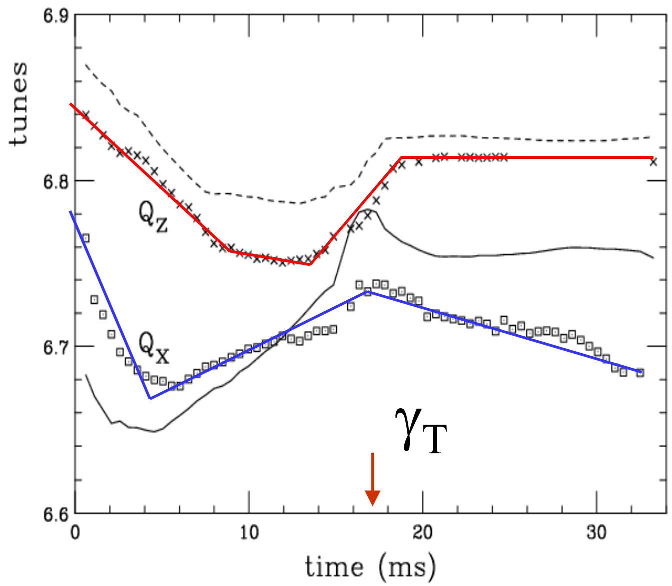
Upshot: Horizontal emittance does not change with beam intensity!

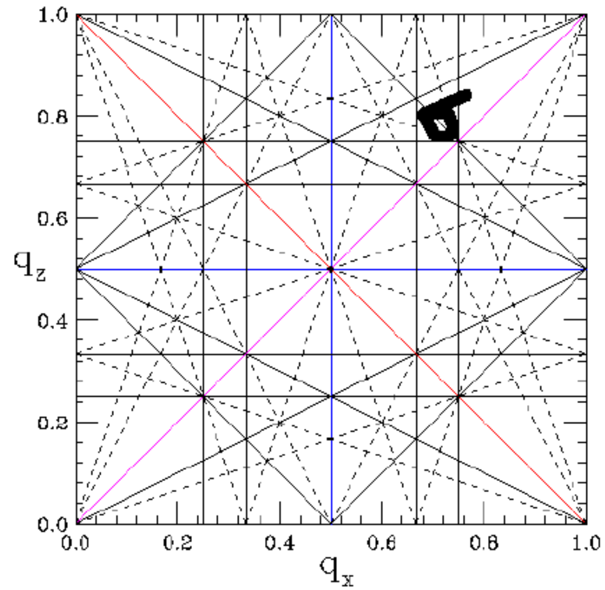
The results are

1. The vertical emittance depends very much on the beam intensity (see right plot)
2. The horizontal emittance is independent of the beam intensity.
3. Detailed data analysis points to the importance of the linear sum and difference resonances at $\nu_x+\nu_z=13$ and $\nu_x-\nu_z=0$.



4. The Montaque resonance at $2\nu_x-2\nu_z=0$ holds the horizontal emittance constant, and causes the growth of the vertical emittance.





An Introduction to ICA*

Three routes toward source signal separation, each makes some assumptions of source signals.

1. Non-gaussian: source signals are assumed to have non-gaussian distribution.

Gaussian pdf
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

2. Non-stationary: source signals have slowly changing power spectra

3. Time correlated: source signals have distinct power spectra.

This is the one we are going to explore

* Often also referred as Blind Source Separation (BSS).

ICA with Time-correlation

- Assumptions

(1) $\langle \mathbf{s}(t)\mathbf{s}(t+\tau)^T \rangle = \mathrm{diag}[\rho_1(\tau), \rho_2(\tau), \ldots, \rho_n(\tau)]$

- Source signals are temporally correlated.
- No overlapping between power spectra of source signals.

As a convention, source signals are normalized, so $\mathbf{C}_s(0) \equiv \langle \mathbf{s}(t)\mathbf{s}(t)^T \rangle = \mathbf{I}$

(2) $\langle \mathbf{n}(t)\mathbf{n}(t+\tau)^T \rangle = \sigma^2 \delta(\tau)\mathbf{I}$

$\langle \mathbf{n}(t)\mathbf{s}(t+\tau)^T \rangle = \mathbf{0}$

Noises are temporally white and spatially decorrelated. And noises are independent from source signals.

In accelerator physics: $\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)$

- Covariance matrix

$$\mathbf{C}_x(0) \equiv \langle \mathbf{x}(t)\mathbf{x}(t)^T \rangle = \mathbf{A}\mathbf{C}_s(0)\mathbf{A}^T + \sigma^2\mathbf{I}$$

$$\mathbf{C}_x(\tau) \equiv \langle \mathbf{x}(t)\mathbf{x}(t+\tau)^T \rangle = \mathbf{A}\mathbf{C}_s(\tau)\mathbf{A}^T, \tau \neq 0$$

So the mixing matrix A is the diagonalizer of the sample covariance matrix C_x. For better performance, mixing matrix A is found as an approximate JOINT diagonalizer of $C_x(\tau)$ with several τ, instead of one. To facilitate the joint diagonalization algorithm and for noise reduction, a two-phase approach is taken.

ICA with Time-correlation

- Algorithm

1. Data whitening

D1,D2 are diagonal

Set to remove noise

$$\mathbf{C}_x(0) = [\mathbf{U}_1, \mathbf{U}_2] \begin{bmatrix} \mathbf{D}_1 & \\ & \mathbf{D}_2 \end{bmatrix} [\mathbf{U}_1, \mathbf{U}_2]^T \quad \text{with} \quad 0 \le \max(\mathbf{D}_2) < \lambda_c \le \min(\mathbf{D}_1)$$

$$\mathbf{z} = \mathbf{D}_1^{-\frac{1}{2}} \mathbf{U}_1^T \mathbf{x} = \mathbf{V}\mathbf{x} \longrightarrow \langle \mathbf{z}\mathbf{z}^T \rangle = \mathbf{I}$$

Benefits of whitening:
1. Reduction of dimension
2. Noise reduction
3. Only rotation (unitary W) is needed to diagonalize.

2. Joint approximate diagonalization

$$\mathbf{C}_z(\tau) = \mathbf{W}\mathbf{C}_s(\tau)\mathbf{W}^T \quad \text{for} \quad \tau = \{\tau_i \mid i = 1, 2, \cdots, k\}$$

3. The mixing matrix A and source signals **s**

$$\mathbf{s} = \mathbf{W}^T \mathbf{V}\mathbf{x}$$

$$\mathbf{A} = (\mathbf{U}_1 \mathbf{D}_1^{\frac{1}{2}})\mathbf{W}$$

Linear Optics Functions Measurements

- The spatial pattern can be used to measure beta function (β), phase advance (ψ) and dispersion (D_x)

1. Betatron function and phase advance

$$x = A_{b1}s_1 + A_{b2}s_2$$

Betatron motion is decomposed to a sine-like signal and a cosine-like signal

$$\beta = a(A_{b1}^2 + A_{b2}^2) \qquad \psi = \tan^{-1}\left(\frac{A_{b1}}{A_{b2}}\right)$$

2. Dispersion

$$x = A_l s_l$$

Orbit shift due to synchrotron oscillation coupled through dispersion

$$D_x = bA_l \qquad \delta = \frac{s_l}{b}$$

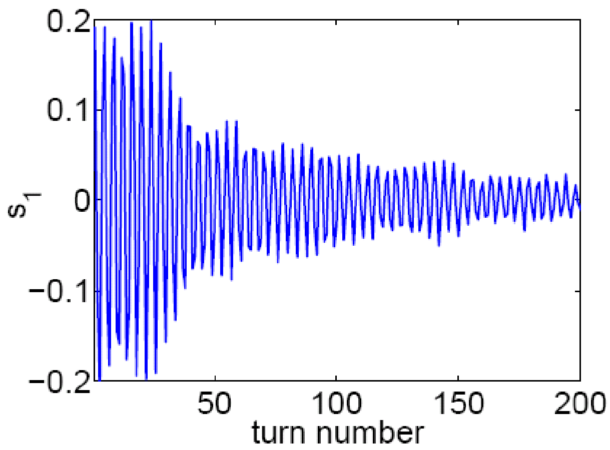
a, *b* are constants to be determined

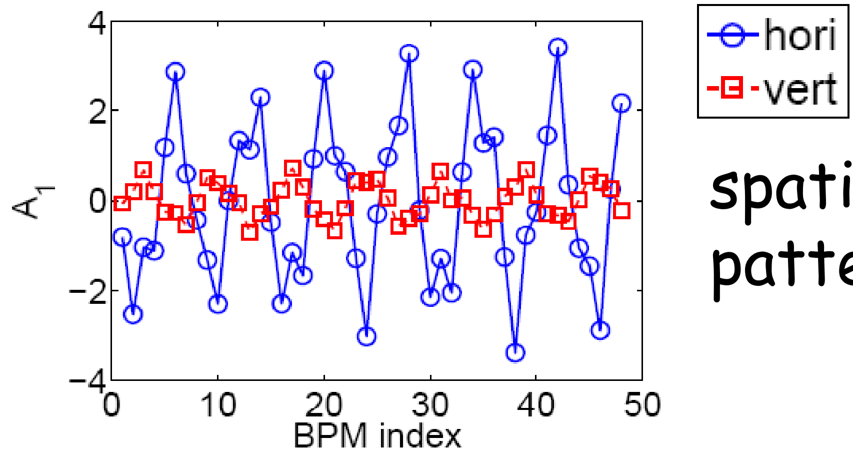
Application to Fermilab Booster

X. Huang et al., PRSTAB 8, 064001 (2005)

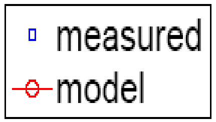
Linear lattice function measurements

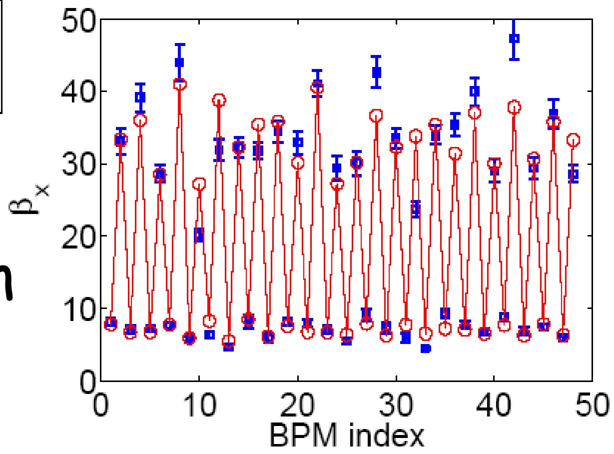
Temporal pattern





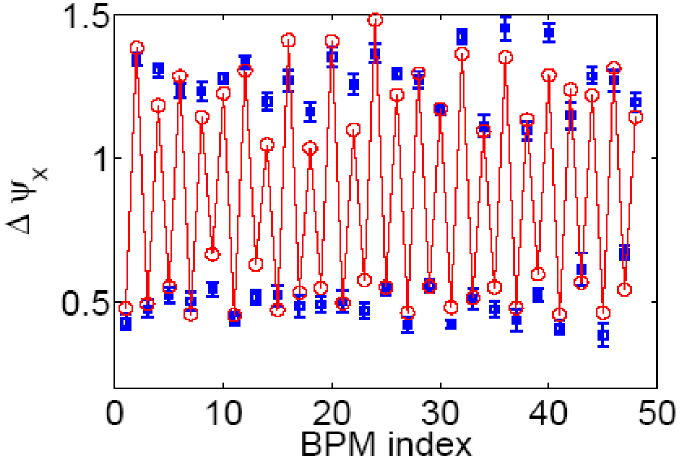
spatial pattern





Beta function

$$\frac{\sigma_\beta}{\beta} = 0.06$$

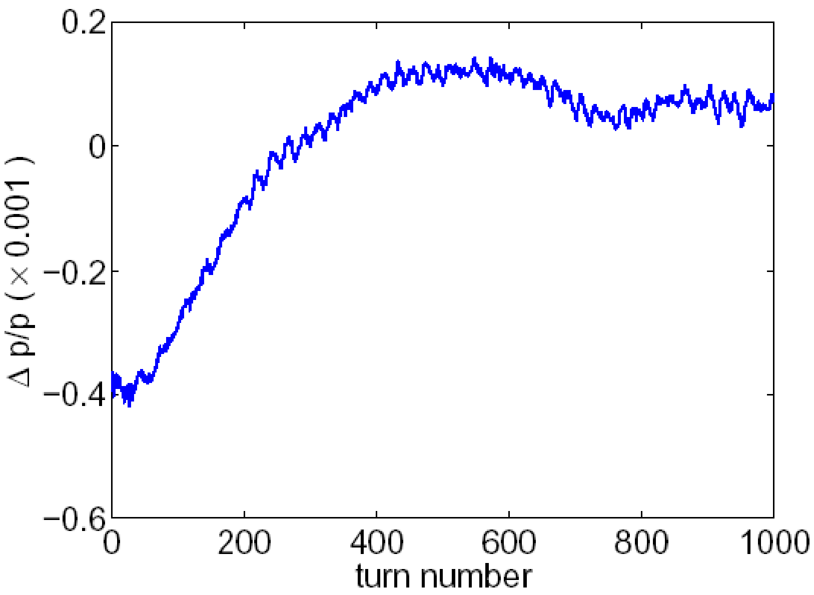


Phase advance

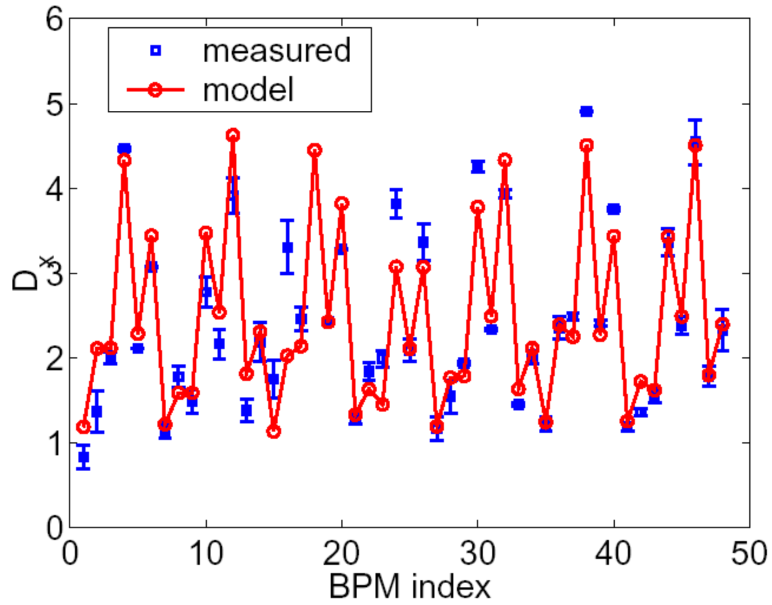
$$\sigma_{\Delta\psi} = 0.03$$

Dispersion Measurement

The dispersion is derived from the mode corresponding to evolution of momentum deviation due to injection energy mismatch (DC beam, one new dogleg).



Injection energy error and its evolution

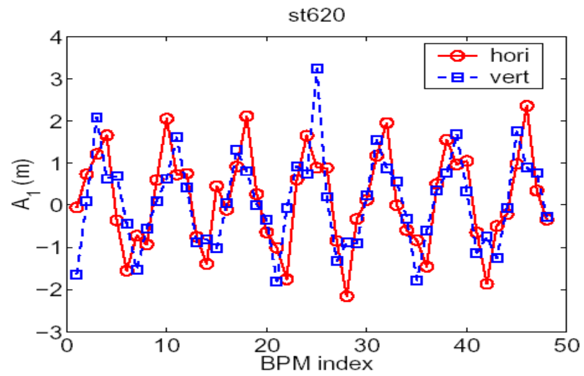


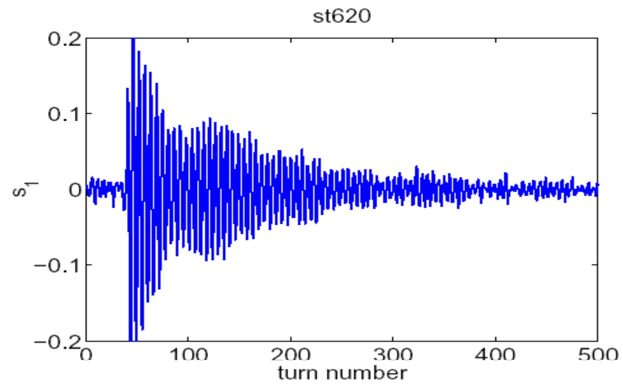
The measured dispersion is compared to model. Error σ_D=0.11 m

Recent Beta Measurements (AC)

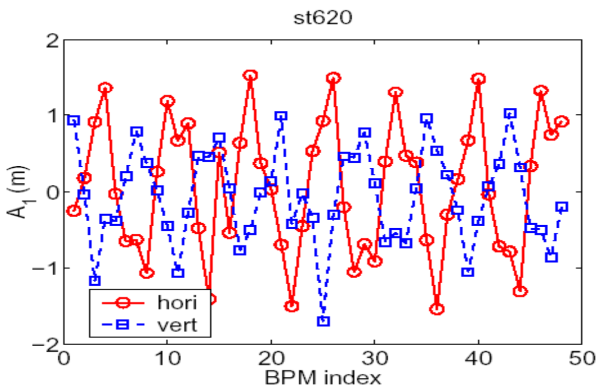
AC beam, with two new doglegs, Pinger frequency 500 Hz (2 ms).

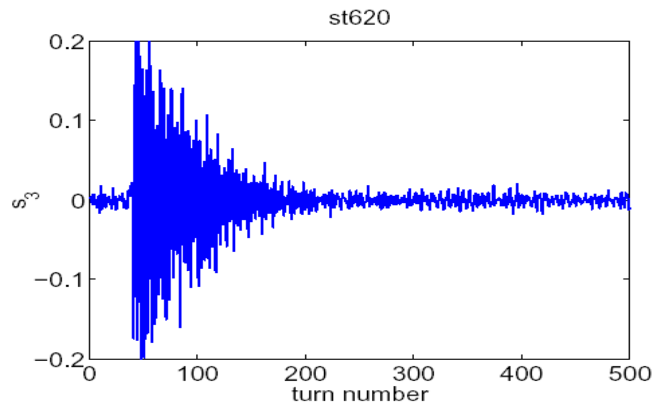
A plus mode





A minus mode



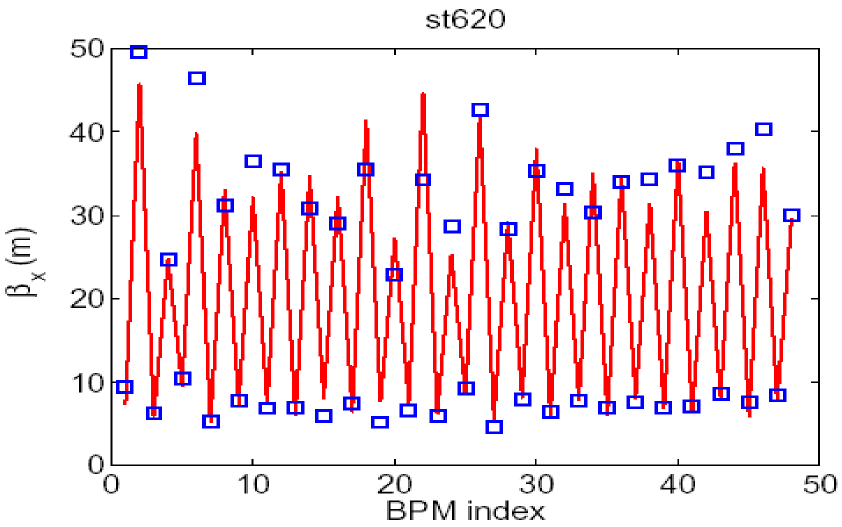


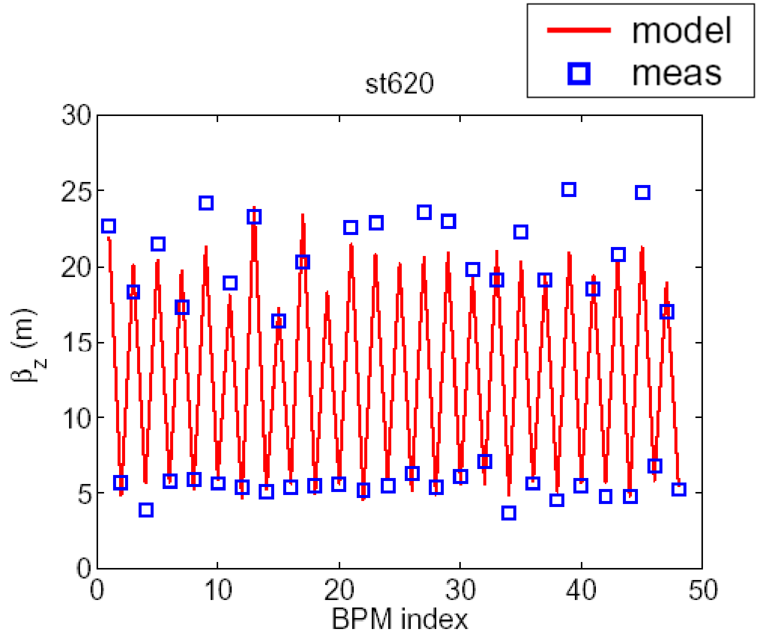
Coupling is strong in ramping cycles.

Beta Function in AC data

The measured beta function compared to model.

Turn 620-670



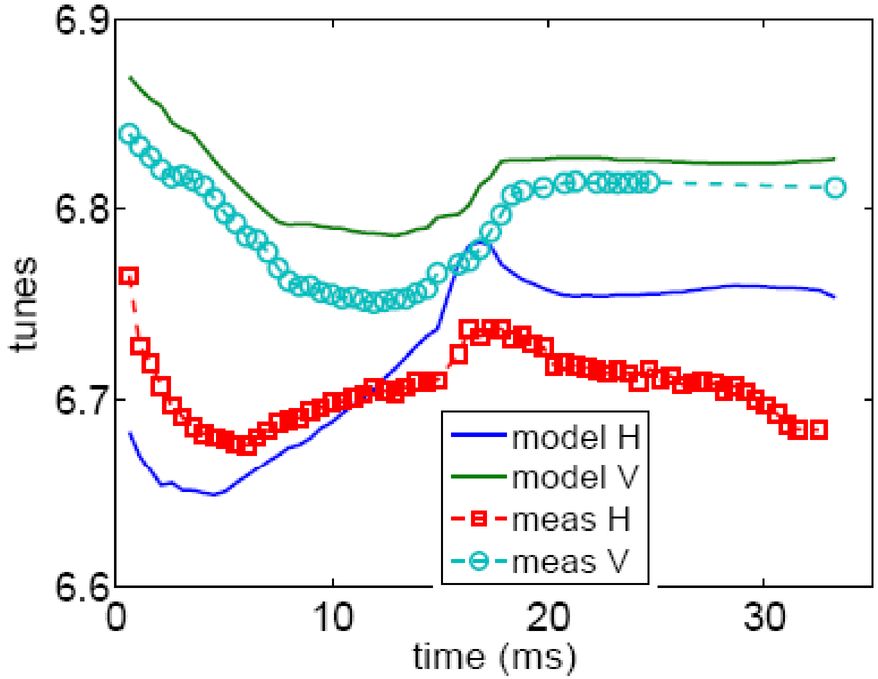


Horizontal beta

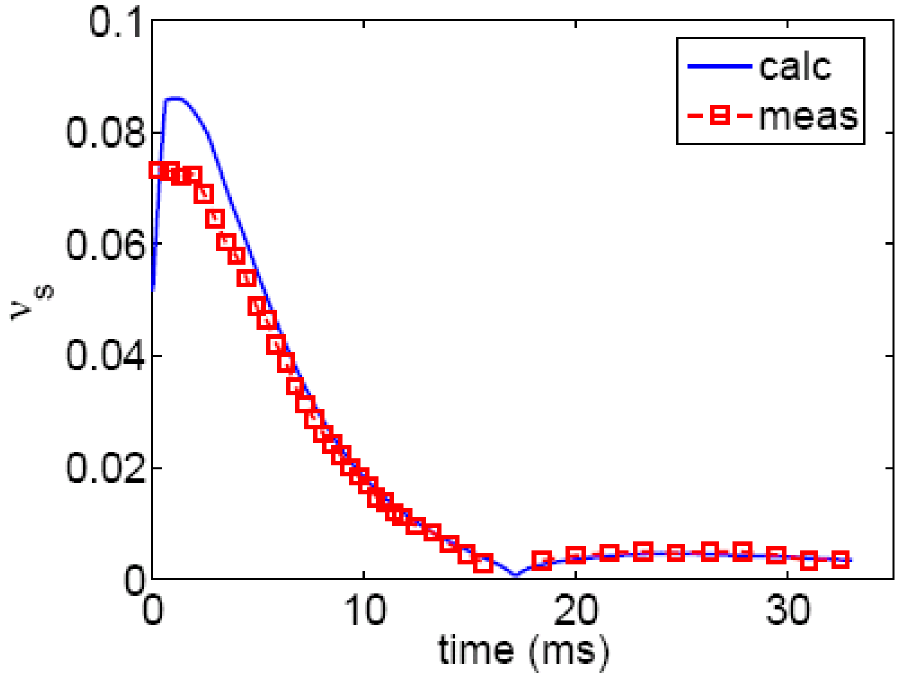
Vertical beta

Phase advance are also measured. Measurements were done in later time of the cycle, too.

Betatron and synchrotron tune measurements



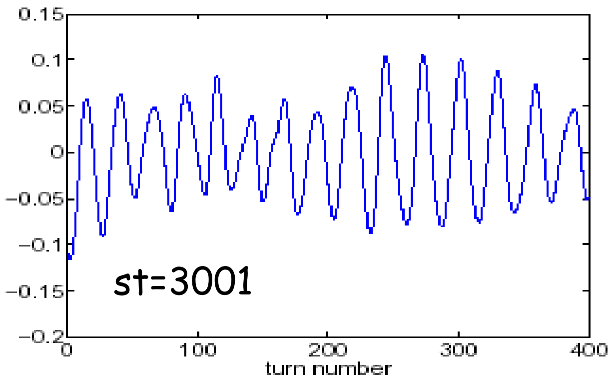
The betatron tunes



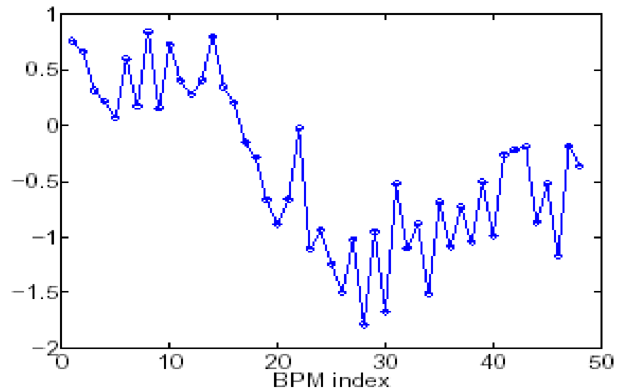
The synchrotron tunes

The clean coherent betatron modes and the interpolated FFT allows betatron tune measurements to high accuracy: <=0.0005 with 250 turns.

Weak Synchrotron Motion

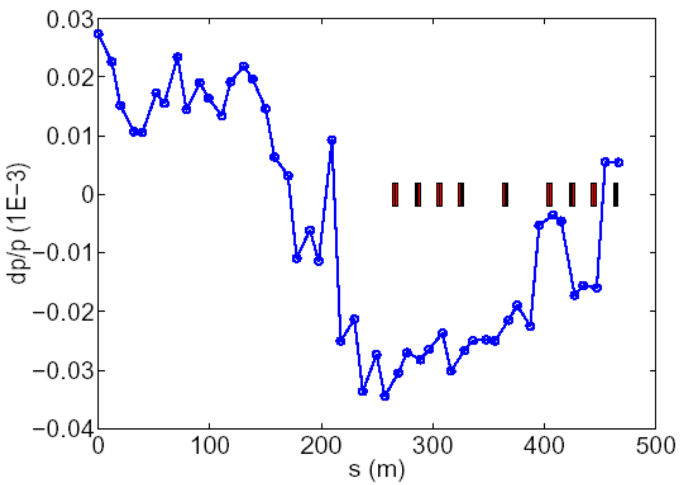


Temporal pattern



Spatial pattern

amp = 0.1 mm

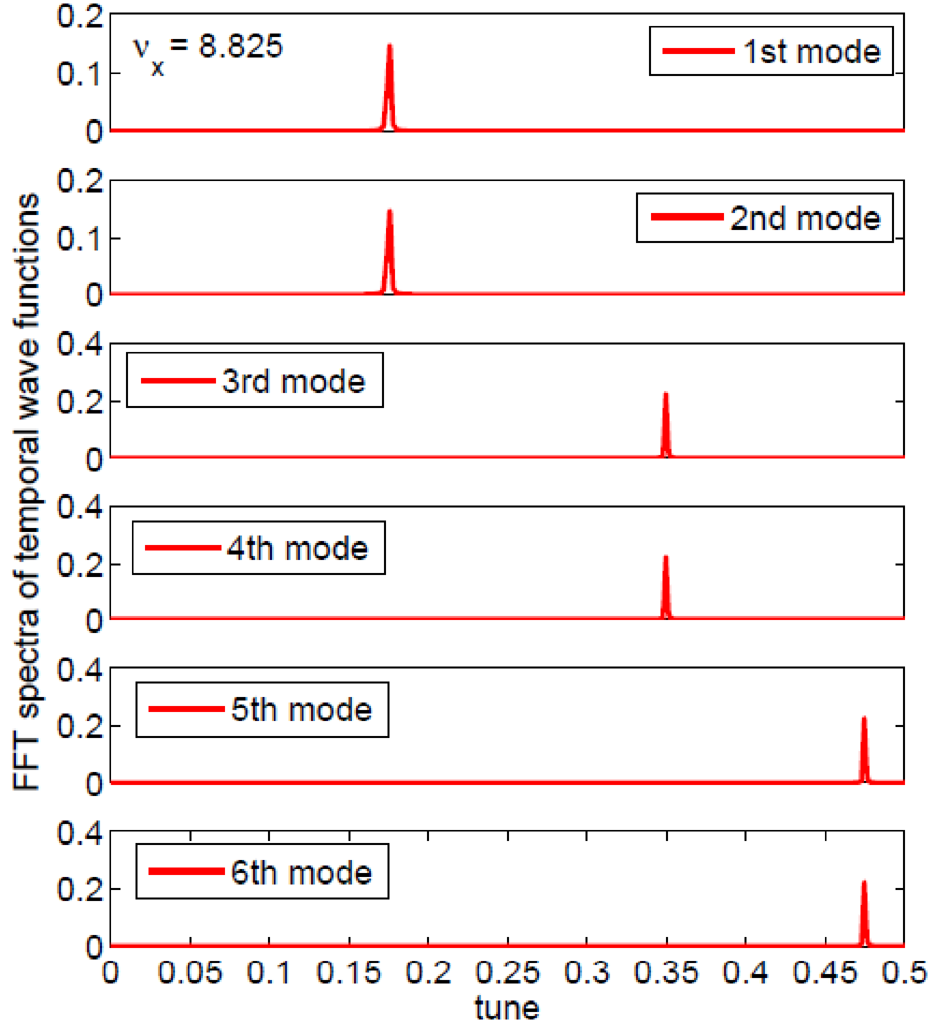


Amplitude of dp/p, spatial pattern normalized by calculated dispersion. Correlated to the RF cavities.

Study of nonlinear motion -- $2\nu_x$ mode

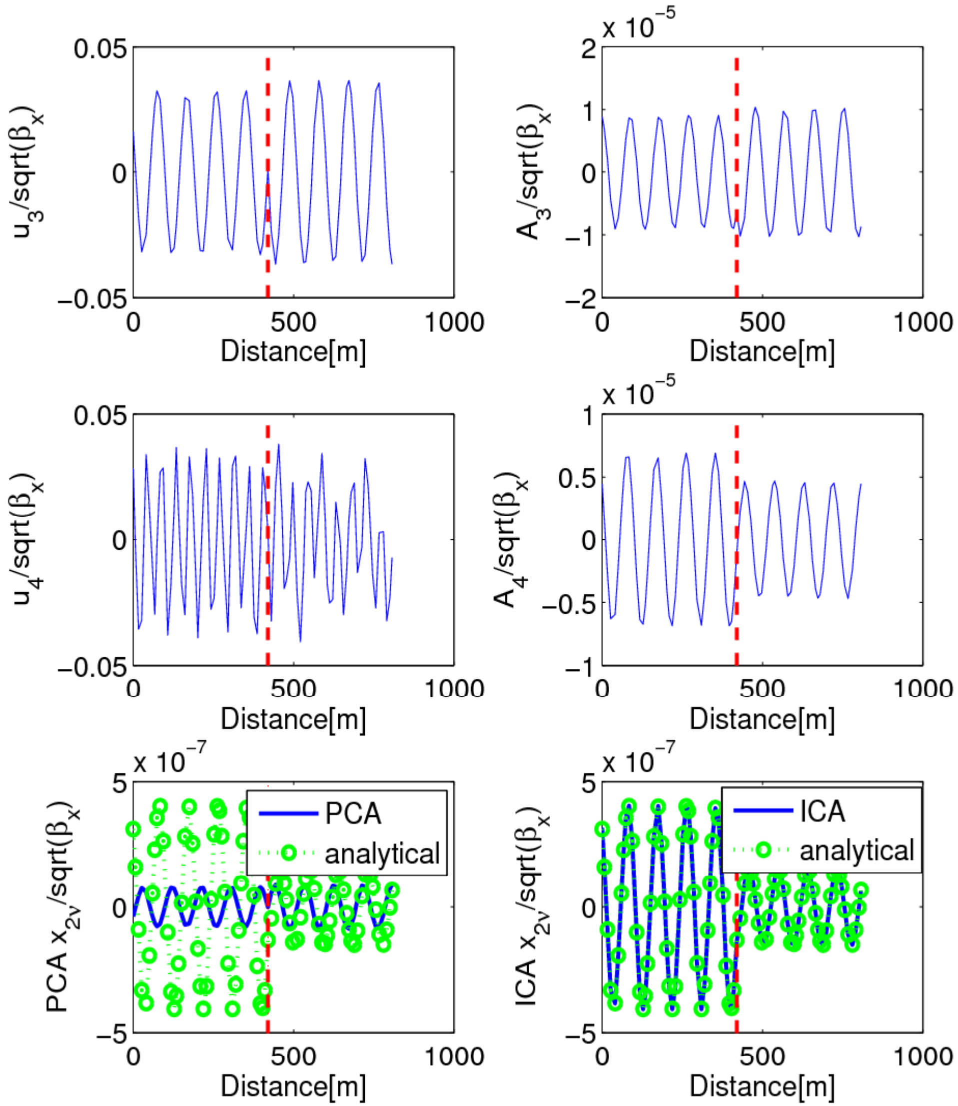
X. Pang and SY Lee, JAP 106,074902 (2009)

- Simple lattice with 12 superperiod of FODO cells
- Add Sextupoles in the lattice.
- Particle tracking was carried out and the data were analyzed by PCA and ICA.
- We found totally 6 important modes. We only consider the 3rd and 4th modes at the tune of $2\nu_x$



The $2\nu_x$ mode

- Compare the spatial function obtained by ICA and PCA
- After ICA processing, the normalized spatial wave functions of the 3rd and 4th modes have simple linear betatron motion outside the sextupole.
- The spatial function of the 4th mode obtained by PCA preprocessing is messy, but still important in a proper ICA analysis.



- Equation of motion of $2\nu_x$ mode
  - Hill's eqn: $$x'' + K_x(s)x = \frac{B_2(s)}{2B\rho}(x^2 - z^2)$$
  - Short sextupole, localized kick $$(B_2/B\rho) = \sum K_2 L \delta(s - s_{\rm sext} - nC)$$
  - Floquet transformation: $$\xi = \frac{x}{\sqrt{\beta_x}} \qquad \phi = \frac{1}{\nu_x}\int \frac{ds}{\beta_x}$$

$$\begin{aligned}
\ddot{\xi} + {\nu_x}^2\xi &= \nu_x^2\beta_x^{3/2}\frac{B_2}{2B\rho}x^2\sum_n \delta(s - s_{\rm sext} - 2n\pi) \\
&= {\nu_x}^2{\beta_x}^{3/2}\frac{B_2}{2B\rho}x^2\sum_n \delta(\phi - \phi_{\rm isext} - 2n\pi)\frac{1}{\nu_x\beta_x} \\
&= \frac{1}{2}\nu_x J_x {\beta_x}^{3/2} K_2 L[\cos(2\nu_x\phi + 2\chi) + 1]\sum_n \delta(\phi - \phi_{\rm isext} - 2n\pi)
\end{aligned}$$

where $\ddot{\xi} = d^2\xi/d\phi^2 \qquad K_2 L = \int B_2(s)ds/B\rho$

- Solution: $\xi = \xi_0 + \xi_1 \quad \xi_0 = x_0/\sqrt{\beta_x} \quad x_0 = \sqrt{2\beta_x J_x}\cos(\nu_x\phi + \chi)$

- Get the particular solution ξ_1

Perturbative solution:

$$\ddot{\xi} + {\nu_x}^2 \xi \; = \; \frac{1}{2} \nu_x J_x {\beta_x}^{3/2} K_2 L [\cos(2\nu_x \phi + 2\chi) + 1] \sum_n \delta(\phi - \phi_{\mathrm{isext}} - 2n\pi)$$

$$\frac{1}{2} \beta_x^{3/2} K_2 L \sum_n \delta(\phi - \phi_{\mathrm{sext}} - 2n\pi) = \sum_k f_k e^{ik\phi}$$

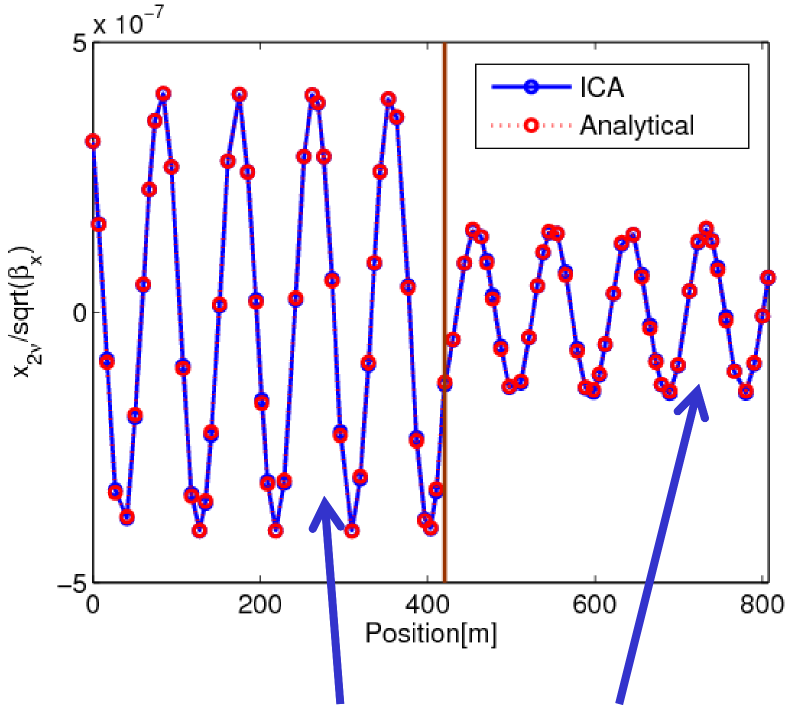
$$f_k = \frac{1}{4\pi} \sum_{\mathrm{sext}} \beta_{x,\mathrm{sext}}^{3/2} K_2 L e^{-ik\phi_{\mathrm{sext}}} \qquad f_k = |f_k| e^{i\varphi_k}$$

$$\ddot{\xi} + {\nu_x}^2 \xi = \sum_k |f_k| \nu_x J_x \cos[(2\nu_x - k)\phi + 2\chi + \varphi_k] + \sum_k |f_k| \nu_x J_x \cos[k\phi + \varphi_k]$$

$$\xi_1 \; = \sum_k \frac{|f_k| \nu_x J_x}{{\nu_x}^2 - (2\nu_x + k)^2} \cos[(2\nu_x + k)\phi + 2\chi + \varphi_k] + \boxed{\sum_k \frac{|f_k| \nu_x J_x}{\nu_x^2 - k^2} \cos[k\phi + \varphi_k]}$$

Closed Orbit

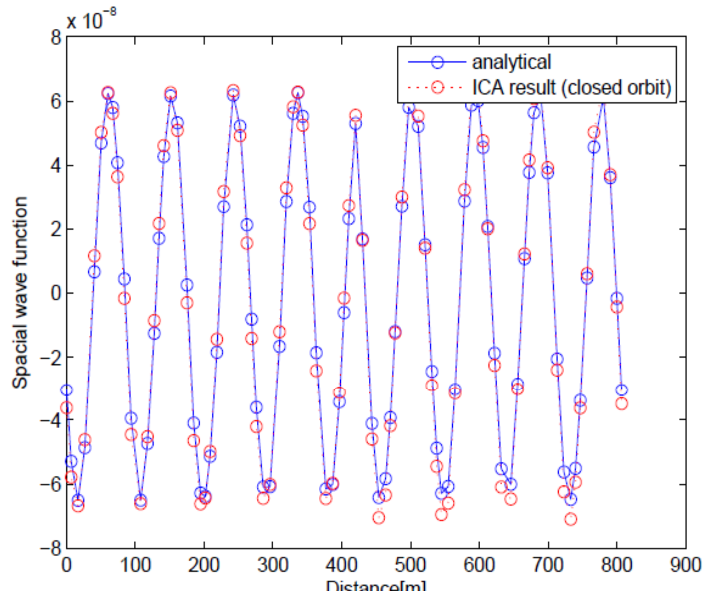
$$x_{2\nu}(s,t) = A_3(s)s_3(t) + A_4(s)s_4(t)$$

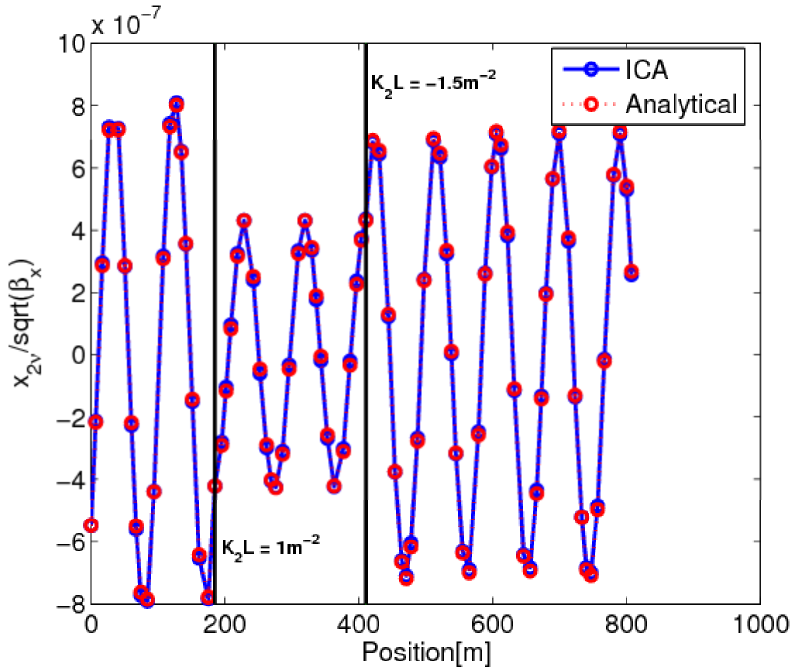


Simple betatron oscillation !

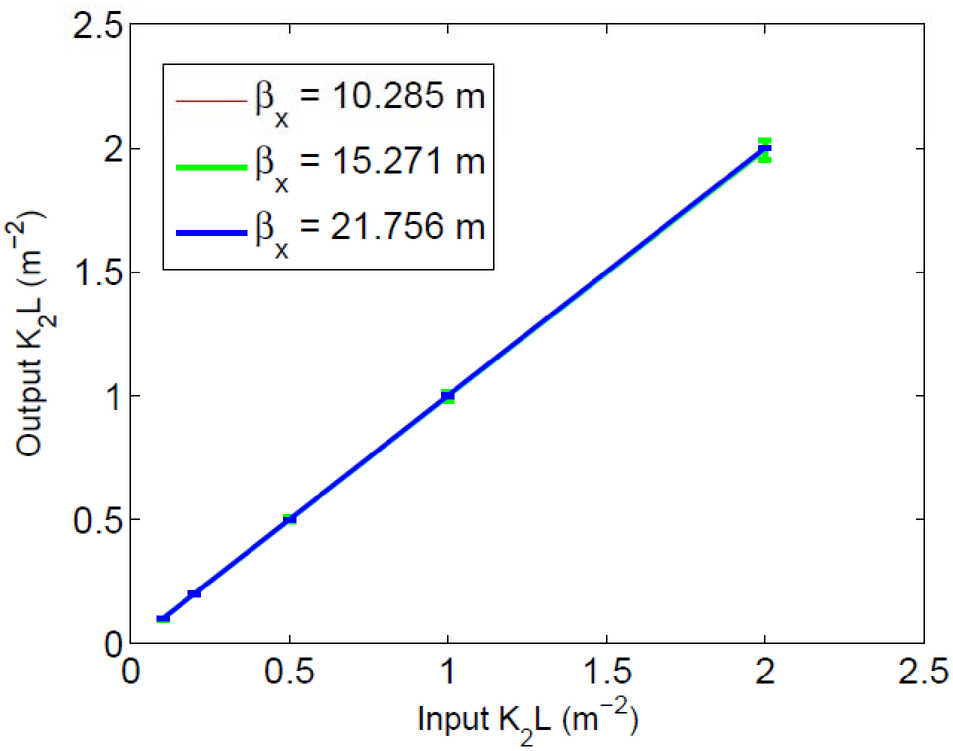
AGS lattice with two sextuples located at 185m and 420.37m, with strength K_2L = $1m^{-2}$ and $-1.5m^{-2}$. Black lines indicate the locations of two sextupoles.

Closed Orbit= x-x_β-$x_{2\nu}$

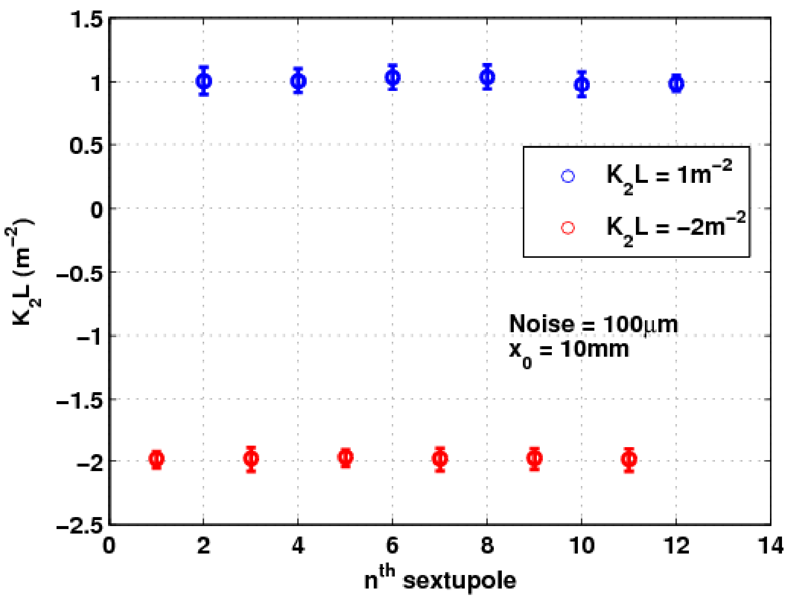


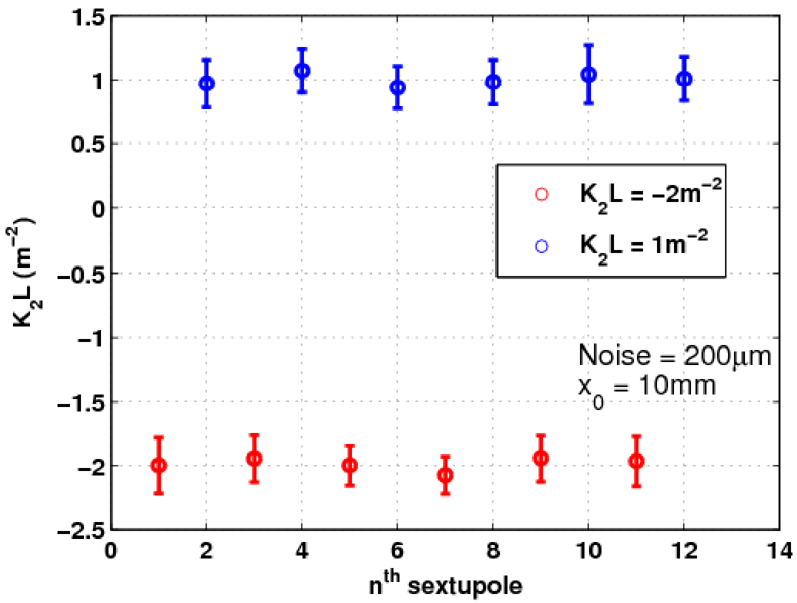


Beam-based measurement of sextupole strengths



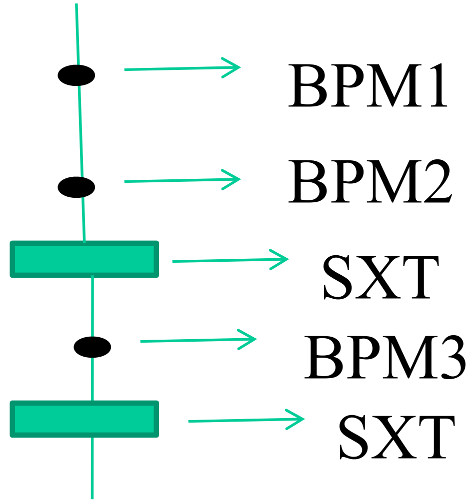
With 12 sextupoles in the lattice





What happens if there is only one BPM between sextupoles?

$$
\begin{aligned}
x_1 &= \sqrt{2\beta_1 J_n} sin[\phi_{1,n} + 2\pi\nu_x(n-1)] \\
x_2 &= \sqrt{2\beta_2 J_n} sin[\phi_{2,n} + 2\pi\nu_x(n-1)] \\
x_3 &= \boxed{\sqrt{2\beta_3 J_n} sin[\phi_{3,n} + 2\pi\nu_x(n-1)]} \rightarrow \hat{x}_3 \\
&+ \sqrt{\beta_s\beta_3} sin[\nu_x(\phi_{3,n} - \phi_{s,n})]\Delta x_s{}'
\end{aligned}
$$



$$
\Delta x_s{}' = \frac{1}{2} K_2 L x_s{}^2 \qquad x_s = \sqrt{2\beta_s J_n} sin[\phi_{s,n} + 2\pi\nu_x(n-1)]
$$

$$
J(n) = \frac{1}{4}\left[\frac{1}{cos^2(\frac{\Delta\phi_{21}}{2})}\left(\frac{x_1}{\sqrt{2\beta_1}} + \frac{x_2}{\sqrt{2\beta_2}}\right)^2 + \frac{1}{sin^2(\frac{\Delta\phi_{21}}{2})}\left(\frac{x_1}{\sqrt{2\beta_1}} - \frac{x_2}{\sqrt{2\beta_2}}\right)^2\right] \qquad \Delta\phi_{21} = \phi_{2,n} - \phi_{1,n}
$$

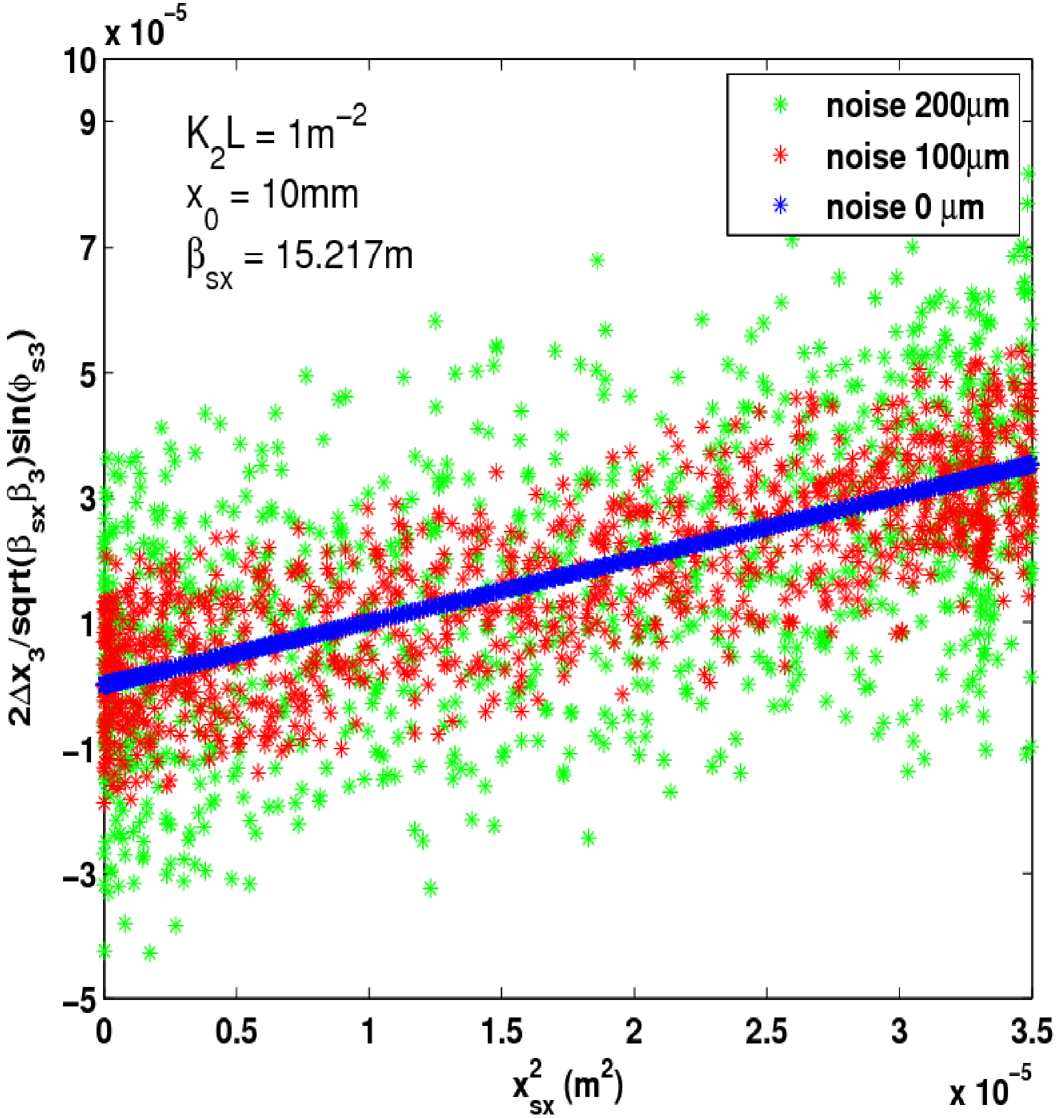
$$
\frac{\hat{x}_3}{\sqrt{2\beta_3}} = \frac{1}{2cos\frac{\Delta\phi_{21}}{2}}\left(\frac{x_1}{\sqrt{2\beta_1}} + \frac{x_2}{\sqrt{2\beta_2}}\right)cos(\Delta\phi_3) - \frac{1}{2sin\frac{\Delta\phi_{21}}{2}}\left(\frac{x_1}{\sqrt{2\beta_1}} - \frac{x_2}{\sqrt{2\beta_2}}\right)sin(\Delta\phi_3) \qquad \phi_3 = \phi_{3,n} - \frac{\phi_{1,n} + \phi_{2,n}}{2}.
$$

$$
\Delta x_3 = x_3 - \hat{x}_3 = \sqrt{\beta_s\beta_3} sin\phi_{s3}\Delta x'
$$

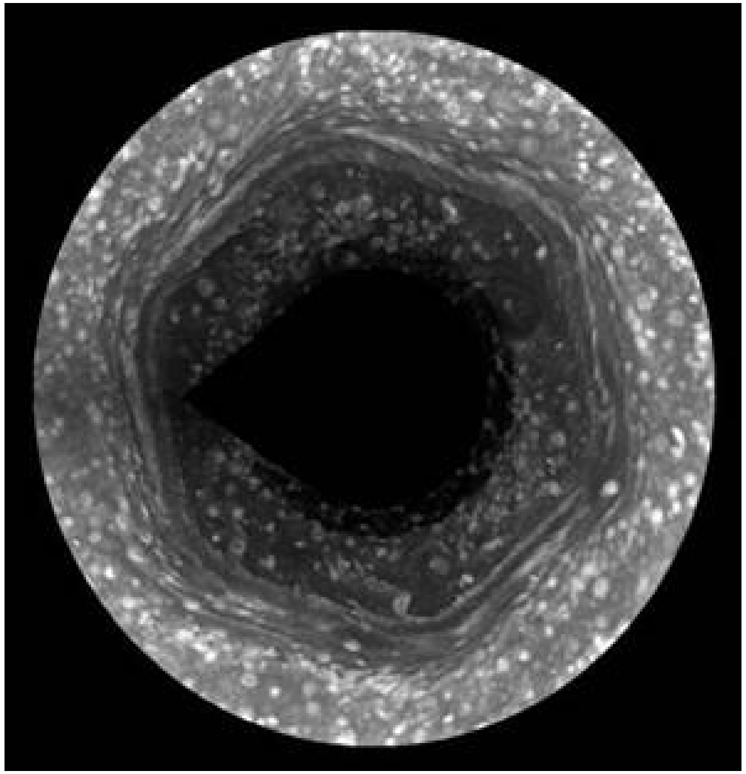
$$
K_2 L = \frac{2\Delta x_3}{\sqrt{\beta_s\beta_3} sin\phi_{s3} x_s{}^2}
$$

$$\frac{2\Delta x_3}{\sqrt{\beta_s \beta_3}\sin\phi_{s3}} = K_2 L x_s^2$$

- With single sextupole in the lattice, very point corresponds to one turn of tracking, totally 1000 turns.
- The slope indicates strength of the sextupole.
- Determine the sextupole strength by finding the slope of the center line of the band.
- The band width is proportional to noise level.
- This method can also be used for other higher order non-linear elements

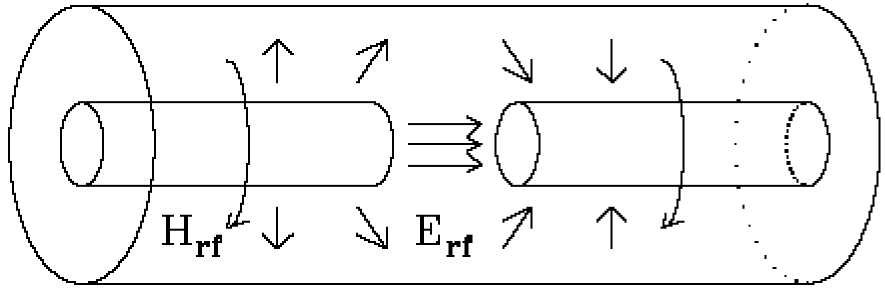


The hexagon, which was discovered by the Voyager spacecraft in the early 1980s, encircles Saturn with an estimated diameter wider than two Earths. The associated jet stream likely whips along the hexagon at about 220 miles per hour (100 meters per second). Cassini has been orbiting Saturn since 2004, and unlike Voyager it has a better angle for viewing the north pole and provides higher-resolution images. But the long darkness of Saturnian winter hid the hexagon from Cassini's visible-light cameras for years. During this time, the craft's infrared instruments were able detect the shape using heat patterns, with the resulting images showing the hexagon is nearly stationary and extends deep into the atmosphere. The images also showed a hotspot and cyclone in the same region.



A mysterious hexagon shape on Saturn, which was captured by cameras aboard NASA's Cassini spacecraft, spans about two Earths and is likely created by the path of a jet stream.

TEM wave coupled cavity:



$$L = \frac{\mu\ell}{2\pi}\ln\frac{r_2}{r_1} + \frac{\mu_c\delta_{\rm skin}\ell}{4\pi}\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \quad C = \frac{2\pi\epsilon\ell}{\ln(r_2/r_1)},$$

$$Z_c = R_c = \sqrt{L/C} = \omega L \approx \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\ \ln\frac{r_2}{r_1}.$$

With exp(jωt) dependence, the current and voltage across the cavity structure is

$$I(s,t) = I_0\cos ks + j(V_0/R_c)\sin ks, \quad V(s,t) = V_0\cos ks + jI_0R_c\sin ks,$$

For a standing wave with shorted end, i.e. V(s=0)=0, we obtain

$$I(s,t) = I_0(t)\cos ks, \quad V(s,t) = +jI_0(t)R_c\sin ks.$$

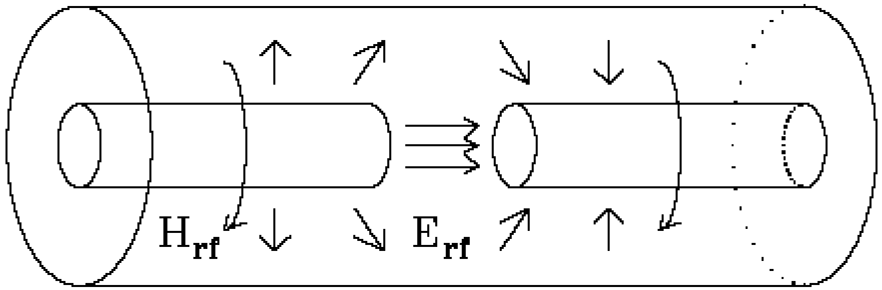
The input impedance of the wave guide is

$$Z_{\rm in} = \frac{V(\ell,t)}{I(\ell,t)} = +jR_c\tan k\ell.$$

The length of the line is chosen to match the input impedance to the reactance of the gap capacitance, i.e.

$$Z_{\rm in} + Z_{\rm gap} = 0, \qquad \tan k\ell_{\rm r} = \frac{1}{\omega R_c C_{\rm gap}} = \frac{1}{g}$$

TEM wave coupled cavity:



The resulting voltage at the gap is

$$V_{\rm rf} = +jI(0)R_{\rm c}\sin k\ell_{\rm r} = +j\frac{I(0)R_{\rm c}}{\sqrt{1+g^2}}.$$

The shunt impedance and Q-factor:

The surface resistivity and the resistance of the transmission line is

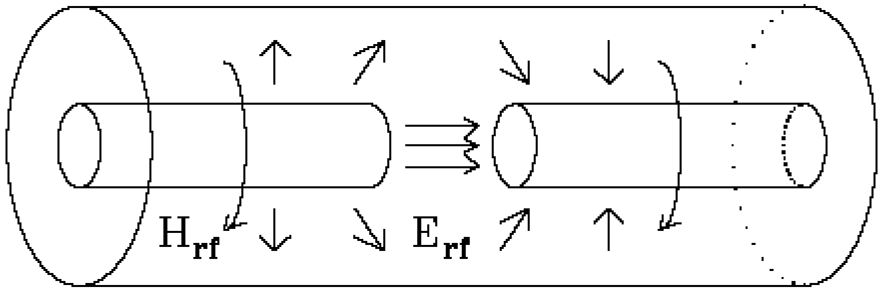
$$R_s = \sqrt{\frac{\mu_c \omega}{2\sigma}}, \qquad R = \frac{R_s \ell}{2\pi}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

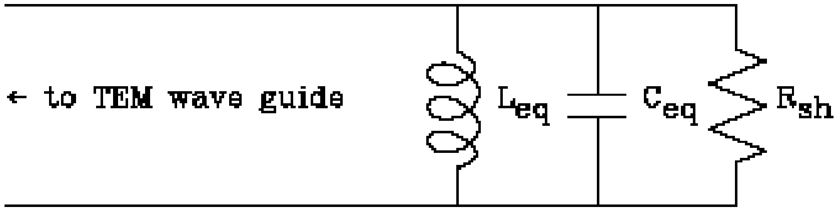
$$\mathrm{Q} = \frac{R_c}{R} = \frac{\omega L}{R} \approx \frac{2r_1 r_2}{(r_1+r_2)\delta_{\rm skin}}\frac{\mu}{\mu_c}\ln\frac{r_2}{r_1}$$

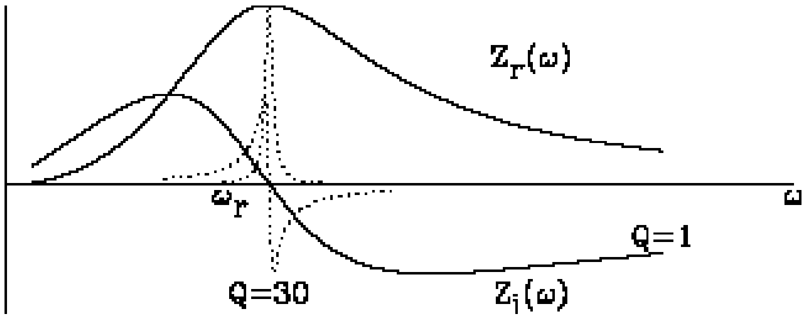
$$P_{\rm d} = \frac{I^2 R}{2}\int_0^{k\ell_r}\cos^2 x\, dx = \frac{I^2 R}{4(1+g^2)}[(1+g^2)\cot^{-1} g + g]$$

$$R_{sh} = \frac{|V_{\rm rf}|^2}{2P_{\rm d}}$$

TEM wave coupled cavity:







$$Z = \left(\frac{1}{R_{\rm sh}} + j\omega C_{\rm eq} + \frac{1}{j\omega L_{\rm eq}}\right)^{-1} = \frac{R_{\rm sh}}{1 + j{\rm Q}(\omega/\omega_{\rm r} - \omega_{\rm r}/\omega)} \approx R_{\rm sh}\cos\psi e^{-j\psi},$$

$$\omega_{\rm r} = (L_{\rm eq}C_{\rm eq})^{-1/2}$$

$${\rm Q} = R_{\rm sh}\sqrt{C_{\rm eq}/L_{\rm eq}}$$

$$\psi = \tan^{-1}\frac{2{\rm Q}(\omega - \omega_{\rm r})}{\omega_{\rm r}}$$

Filling time:

$$Q = \frac{R_c}{R} = \frac{P_{\mathrm{st}}}{P_{\mathrm{d}}} = \frac{\omega W_{\mathrm{s}}}{P_{\mathrm{d}}}, \qquad \frac{dW_{\mathrm{s}}}{dt} = -P_{\mathrm{d}} = -\frac{\omega}{Q} W_{\mathrm{s}}, \qquad \therefore\ W_{\mathrm{s}} = W_{\mathrm{s0}} \exp(-\frac{\omega}{Q} t)$$

The filling time is defined as the time for the electric field or potential to decrease 1/e of its initial value.

$$T_{\mathrm{f}} = \frac{2Q}{\omega}$$